

# MINIMAL MODELS AND ABUNDANCE FOR POSITIVE CHARACTERISTIC LOG SURFACES

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**ABSTRACT.** We discuss the birational geometry of singular surfaces in positive characteristic. More precisely, we establish the minimal model program and the abundance theorem for  $\mathbb{Q}$ -factorial surfaces and for log canonical surfaces. Moreover, in the case where the base field is the algebraic closure of a finite field, we obtain the same results under much weaker assumptions.

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## Part 0. Introduction

In this paper, we consider the minimal model theory for surfaces with some singularities in positive characteristic. If the singularities are  $\mathbb{Q}$ -factorial or log canonical, then we establish the minimal model program and the abundance theorem. In the case where the base field is the algebraic closure of a finite field, we obtain the same result under much weaker assumptions. More precisely, we prove the following two theorems in this paper.

**Theorem 0.1** (Minimal model program). *Let  $X$  be a projective normal surface  $X$ , which is defined over an algebraically closed field  $k$  of positive characteristic. Let  $\Delta$  be an  $\mathbb{R}$ -divisor on  $X$  and let  $\Delta = \sum_{j \in J} \delta_j \Delta_j$  be its prime decomposition. Assume that one of the following conditions holds:*

- (QF)  $X$  is  $\mathbb{Q}$ -factorial and  $0 \leq \delta_j \leq 1$  for all  $j \in J$ .
- (FP)  $k = \overline{\mathbb{F}}_p$  and  $0 \leq \delta_j$  for all  $j \in J$ .
- (LC)  $(X, \Delta)$  is a log canonical surface.

*Then, there exists a sequence of birational morphisms*

$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\phi_0} (X_1, \Delta_1) \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{s-1}} (X_s, \Delta_s) =: (X^\dagger, \Delta^\dagger)$$

*where  $(\phi_{i-1})_*(\Delta_{i-1}) =: \Delta_i$*

*with the following properties.*

- (1) *Each  $X_i$  is a projective normal surface.*
- (2) *Each  $(X_i, \Delta_i)$  satisfies (QF), (FP) or (LC) according as the above assumption.*

(3) For each  $i$ ,  $\text{Ex}(\phi_i) =: C_i$  is a proper irreducible curve such that

$$(K_{X_i} + \Delta_i) \cdot C_i < 0.$$

(4)  $(X^\dagger, \Delta^\dagger)$  satisfies one of the following conditions.

(a)  $K_{X^\dagger} + \Delta^\dagger$  is nef.

(b) There is a surjective morphism  $\mu : X^\dagger \rightarrow Z$  to a smooth projective curve  $Z$  such that  $\mu_* \mathcal{O}_{X^\dagger} = \mathcal{O}_Z$ ,  $-(K_{X^\dagger} + \Delta^\dagger)$  is  $\mu$ -ample and  $\rho(X^\dagger/Z) = 1$ .

(c)  $-(K_{X^\dagger} + \Delta^\dagger)$  is ample and  $\rho(X^\dagger) = 1$ .

In case (a), we say  $(X^\dagger, \Delta^\dagger)$  is a minimal model of  $(X, \Delta)$ .

In case (b) and (c), we say  $(X^\dagger, \Delta^\dagger)$  is a Mori fiber space.

**Theorem 0.2** (Abundance theorem). *Let  $X$  be a projective normal surface  $X$ , which is defined over an algebraically closed field  $k$  of positive characteristic. Let  $\Delta$  be an  $\mathbb{R}$ -divisor on  $X$  and let  $\Delta = \sum_{j \in J} \delta_j \Delta_j$  be its prime decomposition. Assume that one of the following conditions holds:*

(QF)  $X$  is  $\mathbb{Q}$ -factorial and  $0 \leq \delta_j \leq 1$  for all  $j \in J$ .

(FP)  $k = \overline{\mathbb{F}}_p$  and  $0 \leq \delta_j$  for all  $j \in J$ .

(LC)  $(X, \Delta)$  is a log canonical surface.

If  $K_X + \Delta$  is nef, then  $K_X + \Delta$  is semi-ample.

Note that, if  $X$  is a normal surface over  $\overline{\mathbb{F}}_p$ , then  $X$  is  $\mathbb{Q}$ -factorial (cf. Theorem 11.1). In particular,  $K_X + \Delta$  is an  $\mathbb{R}$ -Cartier  $\mathbb{R}$ -divisor.

In the case where the characteristic of the base field is zero, the above two theorems are proven by [Fujino2]. His proofs heavily depend on the Kodaira vanishing theorem and its generalizations. Unfortunately, in positive characteristic, there exist counter-examples to the Kodaira vanishing theorem (cf. [Raynaud]). To prove the above two theorems, we use a result established in [Keel1].

In characteristic zero, the basepoint free theorem follows from the Kawamata–Viehweg vanishing theorem, which is a generalization of the Kodaira vanishing theorem (cf. [Kollár-Mori, Theorem 3.3]). Although we cannot use the Kodaira vanishing theorem, we can show the following basepoint free theorem.

**Theorem 0.3** (Basepoint free theorem). *Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface  $X$ , which is defined over an algebraically closed field  $k$  of positive characteristic. Let  $\Delta$  be a  $\mathbb{Q}$ -divisor. Let  $\Delta = \sum_{j \in J} \delta_j \Delta_j$  be its prime decomposition and assume  $0 \leq \delta_j < 1$  for all  $j \in J$ . Let  $D$  be a nef Cartier divisor satisfying one of the following properties.*

(1)  $D - (K_X + \Delta)$  is nef and big.

(2)  $D - (K_X + \Delta)$  is semi-ample.

Then  $D$  is semi-ample.

Although there exist counter-examples the Kodaira vanishing theorem, we can use the relative Kawamata–Viehweg vanishing theorem for birational morphisms of surfaces by [Kollár-Kovács]. Then, we obtain the following result on rational singularities.

**Theorem 0.4.** *Let  $X$  be a projective normal surface  $X$ , which is defined over an algebraically closed field  $k$  of positive characteristic. Let  $\Delta$  be an  $\mathbb{R}$ -divisor on  $X$ . Let  $\Delta = \sum_{j \in J} \delta_j \Delta_j$  be its prime decomposition and assume  $0 \leq \delta_j \leq 1$  for all  $j \in J$ . Assume that  $X$  has at worst rational singularities. Then, the following assertions hold.*

- (1)  $X$  is  $\mathbb{Q}$ -factorial. In particular, by Theorem 0.1, we can run a  $(K_X + \Delta)$ -minimal model program:

$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\phi_0} (X_1, \Delta_1) \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{s-1}} (X_s, \Delta_s)$$

where  $(\phi_{i-1})_*(\Delta_{i-1}) =: \Delta_i$ .

- (2) Each  $X_i$  has at worst rational singularities.

**0.5** (Overview of related literature). We summarize some literature related to this paper with respect to the surface theory, the minimal model theory and Keel’s result (Theorem 2.2).

(Surface theory) The Italian school established the classification theory for smooth algebraic surfaces, which was generalized by Kodaira, Shafarevich’s seminar and Bombieri–Mumford ([Mumford2], [BMII] and [BMIII]). Theories of log surfaces and normal surfaces have been developed by Iitaka, Kawamata, Miyanishi, Sakai and many others. See, for example, [Sakai] and [Miyanishi]. [Fujita] established the abundance theorem for pairs  $(X, \Delta)$  where  $X$  is a smooth projective surface and  $\Delta$  is a  $\mathbb{Q}$ -boundary, that is,  $\Delta$  is a  $\mathbb{Q}$ -divisor such that, for the prime decomposition  $\Delta = \sum_{j \in J} \delta_j \Delta_j$ , all the coefficients  $\delta_j$  satisfy  $0 \leq \delta_j \leq 1$ . In characteristic zero, [Fujino2] generalized this result. More precisely, [Fujino2] shows that the abundance theorem holds for pairs  $(X, \Delta)$  where  $X$  is a projective normal  $\mathbb{Q}$ -factorial surface and  $\Delta$  is an  $\mathbb{R}$ -boundary. In this paper, we generalize this result to positive characteristic.

(Minimal model theory) In characteristic zero, the minimal model theory has been developed by Kawamata, Kollár, Mori, Shokurov and many others. See, for example, [Kollár-Mori] and [KMM]. To establish fundamental theorems in minimal model theory, we use the Kodaira vanishing theorem and its generalization.

But, in positive characteristic, there exist counter-examples to the Kodaira vanishing theorem even in the case where the dimension is two

(cf. [Raynaud]). In [T], the author established a weak Kodaira vanishing theorem for positive characteristic surfaces and established a basepoint free theorem for klt surfaces. [Kollár] established the contraction theorem for smooth threefolds in positive characteristic. [Kawamata] established the minimal model program for semi-stable threefolds in positive characteristic.

[Fujino2] established the minimal model theory for  $\mathbb{Q}$ -factorial surfaces and log canonical surfaces in characteristic zero. In this paper, we generalize this result to positive characteristic.

(Keel's result) Keel's result (Theorem 2.2) is a key in this paper. Thus, we want to summarize some literature, related to Keel's result. Theorem 2.2 is the surface version of [Keel1, Theorem 0.2]. Keel's proof depends on the Frobenius maps and the theory of the algebraic spaces. Note that Keel's result (Theorem 2.2) holds only in positive characteristic (cf. [Keel1, Section 3]). For alternative proofs of [Keel1, Theorem 0.2], see [CMM] and [FT]. The proofs of [CMM] and [FT] do not depend on the theory of algebraic spaces. [FT] only considers the case of surfaces.

[Keel1] also shows the basepoint free theorem for  $\mathbb{Q}$ -factorial threefolds over  $\overline{\mathbb{F}}_p$  with non-negative Kodaira dimension. Over  $\overline{\mathbb{F}}_p$ , we can often obtain some strong results. The reason is due to Corollary 2.5. See also [Artin], [Keel2], [Mařek] and [Totaro].

**0.6** (Overview of contents). In Part 1, we summarize the notations and two known results: Keel's result (Theorem 2.2) and Fact 2.4, which play crucial roles in this paper.

In Part 2, we prove the case (QF) of Theorem 0.1 and Theorem 0.2. To show the case (QF) of Theorem 0.1, we establish the cone theorem and the contraction theorem. The cone theorem follows from Mori's Bend and Break and the minimal resolution. We consider the Bend and Break for proper normal surfaces in Section 3. The contraction theorem (Theorem 6.2) is obtained by Keel's result (Theorem 2.2).

To show the case (QF) of Theorem 0.2, we divide the argument into the two cases:  $k \neq \overline{\mathbb{F}}_p$  and  $k = \overline{\mathbb{F}}_p$ . Since we treat the case  $k = \overline{\mathbb{F}}_p$  in Part 3, we prove Theorem 0.2 only for the case  $k \neq \overline{\mathbb{F}}_p$ . By a standard argument, we may assume that  $\Delta$  is a  $\mathbb{Q}$ -divisor (Section 9). First, we prove  $\kappa := \kappa(X, K_X + \Delta) \geq 0$  (Theorem 8.1). This follows from the same argument as [Fujino2]. Second, we consider the three cases:  $\kappa = 0$ ,  $\kappa = 1$  and  $\kappa = 2$ . If  $\kappa = 1$ , then the assertion follows from a more general known result (Proposition 6.4). By using Keel's result and the contraction theorem, we can prove the case of  $\kappa = 2$  (Proposition 7.2). In the case where  $\kappa = 0$  (Theorem 8.5), we use the

arguments in [Fujino2] and [Fujita], which depends on the classification of smooth surfaces.

In Part 3, we prove the case (FP) of Theorem 0.1 and Theorem 0.2. The case (FP) of Theorem 0.1 follows from Keel's result (Theorem 2.2) and Corollary 2.5. The proof of the case (FP) of Theorem 0.2 is almost all the same as [Mařek]. In Part 3, we also show that normal surfaces over  $\overline{\mathbb{F}}_p$  are  $\mathbb{Q}$ -factorial (Section 12).

In Part 4, we consider the case (LC) of Theorem 0.1 and Theorem 0.2. To show the case (LC) of Theorem 0.1, we describe the log canonical surface singularities by using results obtained in Part 2. This is discussed in Section 14. The case (LC) of Theorem 0.2 follows from a known result (cf. [Fujita]).

In Part 5, we generalize the results in Part 2, Part 3 and Part 4 to the relative situations. The relative version of Theorem 0.1 and Theorem 0.2 are Theorem 17.2 and Corollary 18.5, respectively.

In Part 6, we prove Theorem 0.3 and Theorem 0.4. Note that, in characteristic zero, the basepoint free theorem holds for log canonical varieties (cf. [Fujino1, Theorem 13.1]). Its proof heavily depends on the Kodaira vanishing theorem and its generalizations. Although, in positive characteristic, there exist counter-examples to the Kodaira vanishing theorem (cf. [Raynaud]), we can establish the basepoint free theorem for surfaces (Theorem 0.3). Our proof depends on the Keel's result (Theorem 2.2), the classification of smooth surfaces and the Riemann–Roch theorem.

Theorem 0.4 follows from the relative Kawamata–Viehweg vanishing theorem for birational morphisms of surfaces.

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## Part 1. Notations and known results

### 1. NOTATIONS

**1.1 (Notations).** We will freely use the notation and terminology in [Kollár-Mori].

We will not distinguish the notations line bundles, invertible sheaves and Cartier divisors. For example, we will write  $L + M$  for line bundles  $L$  and  $M$ .

Throughout this paper, we work over an algebraically closed field  $k$ , whose characteristic  $\text{char } k =: p$  supposed to be positive unless otherwise mentioned.

In this paper, a *variety* means an integral scheme which is separated and of finite type over  $k$ . A *curve* or a *surface* means a variety whose dimension is one or two, respectively.

Let  $D$  be an  $\mathbb{R}$ -divisor and let  $D = \sum_{j \in J} d_j D_j$  be its prime decomposition. For a real number  $a$ , we define  $D \geq a$  by  $d_j \geq a$  for all  $j \in J$ . We define  $D \leq a$ ,  $D > a$  and  $D < a$  in the same way.

We say  $D$  is an  $\mathbb{R}$ -*boundary* (resp. a  $\mathbb{Q}$ -*boundary*) if  $D$  is an  $\mathbb{R}$ -divisor (resp. a  $\mathbb{Q}$ -divisor) and  $0 \leq D \leq 1$ .

**Definition 1.2** (Semi-ample  $\mathbb{R}$ -divisors). Let  $\pi : X \rightarrow S$  be a proper morphism between varieties. Let  $D$  be an  $\mathbb{R}$ -Cartier  $\mathbb{R}$ -divisor. We say  $D$  is  $\pi$ -*semi-ample* if

$$D = \sum_{1 \leq i \leq N} d_i D_i$$

where  $d_i \in \mathbb{R}_{\geq 0}$  and  $D_i$  is a  $\pi$ -semi-ample Cartier divisor for every  $i$ . For more details, see [Fujino1, Section 4].

## 2. KNOWN RESULTS

**2.1** (Keel's result). In positive characteristic, we can not use the Kodaira vanishing theorem. But we can use the following theorem by Keel. The following assertion is the surface version of the original theorem by Keel. For an alternative proof, see [CMM] and [FT, Section 2].

**Theorem 2.2** (Keel's result). *Let  $X$  be a projective normal surface over an algebraically closed field  $k$  of positive characteristic. Let  $L$  be a nef and big line bundle. Let  $E(L)$  be the reduced subscheme whose support is the union of all the curves  $C$  with  $L \cdot C = 0$ . Then,  $L$  is semi-ample iff  $L|_{E(L)}$  is semi-ample.*

*Proof.* See [Keel1, Theorem 0.2]. □

**2.3** (Difference between  $k \neq \overline{\mathbb{F}}_p$  and  $k = \overline{\mathbb{F}}_p$ ). In this paper, we often divide the argument into the two cases: (1)  $k \neq \overline{\mathbb{F}}_p$  and (2)  $k = \overline{\mathbb{F}}_p$ . The reason for this comes from the following fact.

**Fact 2.4.** *Let  $k$  be an algebraically closed field of arbitrary characteristic.*

- (1) *If  $k \neq \overline{\mathbb{F}}_p$ , all abelian varieties over  $k$  have infinite rank.*
- (2) *If  $k = \overline{\mathbb{F}}_p$ , all group schemes of finite type over  $\overline{\mathbb{F}}_p$  are torsion groups.*

*Proof.* (1) See [Frey-Jarden, Theorem 10.1].

(2) Let  $X$  be a group scheme of finite type over  $\overline{\mathbb{F}}_p$ . Let  $P$  be a closed point of  $X$ . Then we see that  $P$  and  $X$  are defined over a finite field. Thus we can consider  $P$  as a rational point of a group scheme of finite type over a finite field. Since this group is finite,  $P$  is a torsion.  $\square$

As a corollary, we obtain the following information on the line bundles of varieties over  $\overline{\mathbb{F}}_p$ .

**Corollary 2.5.** *Let  $X$  be a projective variety over  $\overline{\mathbb{F}}_p$  and let  $D$  be a Cartier divisor. If  $D \equiv 0$ , then  $D$  is a torsion in  $\text{Pic } X$ .*

*Proof.* Consider the Picard space of  $X$  and apply Fact 2.4. For more details, see [Keel1, Lemma 2.16].  $\square$

## Part 2. $\mathbb{Q}$ -factorial surfaces

### 3. BEND AND BREAK

In this section, we consider Mori's Bend-and-Break for proper normal surfaces. We use the following intersection theory for normal surfaces by [Mumford1].

**Definition 3.1** (Intersection theory by Mumford). Let  $X$  be a normal surface and let  $f : X' \rightarrow X$  be a resolution of singularities. Let  $E_1, \dots, E_n$  be the exceptional curves of  $f$ . Let  $C$  be a proper curve in  $X$  and let  $D$  be an  $\mathbb{R}$ -divisor on  $X$ . Let  $C'$  and  $D'$  be their proper transforms respectively.

- (1) We define  $f^*D := D' + \sum e_i E_i$  where all  $e_i$  are real numbers uniquely determined by the linear equations  $(D' + \sum e_i E_i) \cdot E_j = 0$  for  $j = 1, \dots, n$ . Note that the intersection matrix  $(E_i \cdot E_j)$  is negative definite (cf. [Kollár-Mori, Lemma 3.40]).
- (2) We define the intersection pairing by  $C \cdot D := f^*C \cdot f^*D = C' \cdot f^*D$ .
- (3) If  $X$  is proper, then we can naturally extend this intersection theory to Weil divisors with  $\mathbb{Q}$  or  $\mathbb{R}$  coefficients by linearity.

**Definition 3.2.** Let  $X$  be a proper normal surface and let  $D$  and  $D'$  be  $\mathbb{R}$ -divisors. We denote  $D \equiv_{\text{Mum}} D'$  if  $D \cdot C = D' \cdot C$  for every curve  $C$  in  $X$ .

Let us define the Bend-and-Break. This is the key for the proof of the cone theorem.

**Definition 3.3** (Bend-and-Break). Let  $X$  be a proper normal surface. We say  $X$  satisfies *Bend-and-Break* if  $X$  satisfies the following two conditions (BBI) and (BBII):



- (BBI) If  $Z$  is a rational curve in  $X$ , then  $Z \equiv_{\text{Mum}} Z_1 + \cdots + Z_r$ , where each  $Z_i$  is a rational curve and  $-Z_i \cdot K_X \leq 3$ .
- (BBII) Let  $C$  be a curve in  $X$  with  $C \cdot K_X < 0$ . Then for an arbitrary point  $c_0 \in C \setminus \text{Sing } X$ , there exists a positive integer  $n(X, C, c_0)$  with the following conditions. For an arbitrary positive integer  $n$  with  $n \geq n(X, C, c_0)$ , there exist a non-negative integer  $\alpha_n$ , a curve  $C_n$  and an effective 1-cycle  $Z_n$  with the following four conditions:
- (a)  $p^n C \equiv_{\text{Mum}} \alpha_n C_n + Z_n$ .
  - (b)  $Z_n = Z_{n,1} + \cdots + Z_{n,r_n}$ , where each  $Z_{n,i}$  is a rational curve.
  - (c)  $-\alpha_n C_n \cdot K_X \leq 2g(C_{\text{normal}})$  where  $C_{\text{normal}}$  is the normalization of  $C$ .
  - (d)  $c_0 \in \text{Supp } Z_n$ .

The smooth case is the original Bend-and-Break proven by Mori.

**Proposition 3.4.** *If  $X$  is a projective smooth surface. then  $X$  satisfies Bend-and-Break.*

*Proof.* See [Mori1, Theorem 4 and Theorem 5] and its proofs.  $\square$

Using this result, we extend the Bend-and-Break to the proper normal surfaces.

**Proposition 3.5.** *If  $X$  is a proper normal surface, then  $X$  satisfies Bend-and-Break.*

*Proof.* Let  $f : X' \rightarrow X$  be the minimal resolution and  $K_{X'} + \sum e_i E_i = f^*(K_X)$  where all  $E_i$  are exceptional curves and  $e_i \geq 0$ .

(BBI): Let  $Z$  be a rational curve in  $X$  and let  $Z'$  be its proper transform. Here,  $Z'$  is rational. Since  $X'$  is smooth,  $X'$  satisfies (BBI). Therefore,  $Z' \equiv Z'_1 + \cdots + Z'_r$  and all  $Z'_i$  are rational curves and  $-Z'_i \cdot K_{X'} \leq 3$ . Apply  $f_*$  to this equation. We obtain that  $Z \equiv_{\text{Mum}} Z_1 + \cdots + Z_r$ , where  $f_* Z'_i = Z_i$ . Note that  $Z_i$  may be zero. But if all of  $Z_i$  are zero, then we have  $Z \equiv_{\text{Mum}} 0$ . This is a contradiction. Moreover, the above relation between  $K_X$  and  $K_{X'}$  shows that  $-Z_i \cdot K_X \leq 3$ .

(BBII): Let  $C$  be a curve in  $X$  with  $C \cdot K_X < 0$  and let  $C'$  be its proper transform. We see  $C' \cdot K_{X'} < 0$  from the above relation between canonical divisors. Let  $c_0$  be an arbitrary element of  $C \setminus \text{Sing } X$  and let  $c'_0$  be a point of  $C'$  such that  $f(c'_0) = c_0$ . Since  $X'$  is smooth,  $X'$  satisfies (BBII). Thus we obtain  $n(X', C', c'_0)$ ,  $\alpha'_n$ ,  $C'_n$  and  $Z'_n$ . Let  $n(X, C, c_0) := n(X', C', c'_0)$ ,  $\alpha_n C_n := f_*(\alpha'_n C'_n)$  and  $Z_n := f_*(Z'_n)$ . It is easy to see that these satisfy (BBII).  $\square$

From now on, let us generalize this result for pairs  $(X, \Delta)$ .

**Definition 3.6** ( $(K_X + \Delta)$ -Bend-and-Break). Let  $X$  be a proper normal surface and let  $\Delta$  be an effective  $\mathbb{R}$ -divisor. Let  $\Delta = \sum b_i B_i$  be its prime decomposition. We say  $(X, \Delta)$  satisfies  $(K_X + \Delta)$ -Bend-and-Break if  $X$  and  $\Delta$  satisfy the following two conditions (BBI) and (BBII):

- (BBI) There exists a positive integer  $L(X, \Delta)$  satisfying (1) and (2).  
 (1) If  $Z$  is a rational curve in  $X$ , then  $Z \equiv_{\text{Mum}} Z_1 + \cdots + Z_r$ , where all  $Z_i$  are rational curves and  $-Z_i \cdot (K_X + \Delta) \leq L(X, \Delta)$ .  
 (2) If  $B_i^2 < 0$ , then  $-B_i \cdot (K_X + \Delta) \leq L(X, \Delta)$ .  
 (BBII) Let  $C$  be a curve in  $X$  with  $C \cdot (K_X + \Delta) < 0$  and  $C \neq B_i$  for all  $i$  such that  $B_i^2 < 0$ . Then, for an arbitrary point  $c_0 \in C \setminus \text{Sing} X$ , there exists a positive integer  $n(X, \Delta, C, c_0)$  with the following conditions. For an arbitrary integer  $n$  with  $n \geq n(X, \Delta, C, c_0)$ , there exist a non-negative integer  $\alpha_n$ , a curve  $C_n$  and an effective 1-cycle  $Z_n$  with the following four conditions:  
 (a)  $p^n C \equiv_{\text{Mum}} \alpha_n C_n + Z_n$ .  
 (b)  $Z_n = Z_{n,1} + \cdots + Z_{n,r_n}$ , where all  $Z_{n,i}$  are rational curves.  
 (c)  $-\alpha_n C_n \cdot (K_X + \Delta) \leq 2g(C_{\text{normal}})$  where  $C_{\text{normal}}$  is the normalization of  $C$  or  $C_n = B_i$  for some  $i$  such that  $B_i^2 < 0$ .  
 (d)  $c_0 \in \text{Supp} Z_n$ .

We obtain the following main result in this section.

**Theorem 3.7.** *If  $X$  is a proper normal surface and  $\Delta$  is an effective  $\mathbb{R}$ -divisor, then  $(X, \Delta)$  satisfies  $(K_X + \Delta)$ -Bend-and-Break.*

*Proof.* We write the prime decomposition  $\Delta = \sum b_i B_i$ .

(BBI): Let

$$L(X, \Delta) := \max(\{3\} \cup \{-(K_X + \Delta) \cdot B_\mu\})$$

where  $B_\mu$  ranges over the prime components of  $\Delta$  with  $B_\mu^2 < 0$ . We check the conditions (1) and (2). But, (2) is obvious. Thus, let us prove (1). Let  $Z$  be a rational curve in  $X$ . By Proposition 3.5, we have  $Z \equiv_{\text{Mum}} Z_1 + \cdots + Z_r$  where any  $Z_j$  is rational and satisfies  $-Z_j \cdot K_X \leq 3$ . If  $Z_j = B_\mu$  with  $B_\mu^2 < 0$ , then we obtain  $-Z_j \cdot (K_X + \Delta) \leq L(X, \Delta)$ . If  $Z_j \neq B_\mu$ , then we have

$$-Z_j \cdot (K_X + \Delta) \leq -Z_j \cdot K_X \leq 3 \leq L(X, \Delta).$$

(BBII): Let  $C$  be a curve in  $X$  with  $C \cdot (K_X + \Delta) < 0$  and with  $C \neq B_i$  for all  $B_i$  such that  $B_i^2 < 0$ . Then we obtain the following inequalities:

$$C \cdot K_X \leq C \cdot (K_X + \Delta) < 0.$$

By Proposition 3.5, we can use the Bend-and-Break in the sense of Definition 3.3. Let

$$n(X, \Delta, C, c_0) := n(X, C, c_0).$$

These satisfies the four conditions of (BBII) of Definition 3.6. Indeed, (a)(b)(d) are obvious. We consider (c). If  $C_n \neq B_i$  for all  $B_i$  such that  $B_i^2 < 0$ , then we have

$$-\alpha_n C_n \cdot (K_X + \Delta) \leq -\alpha_n C_n \cdot K_X \leq 2g(C_{\text{normal}}).$$

This completes the proof.  $\square$

Let us calculate  $L(X, \Delta)$  in the case where  $\Delta$  is an  $\mathbb{R}$ -boundary.

**Proposition 3.8.** *Let  $X$  be a proper normal surface and let  $\Delta$  be an  $\mathbb{R}$ -boundary. Then,  $(X, \Delta)$  satisfies  $(K_X + \Delta)$ -Bend-and-Break for  $L(X, \Delta) = 3$ .*

*Proof.* By the proof of Theorem 3.7,  $(X, \Delta)$  satisfies  $(K_X + \Delta)$ -Bend and Break for

$$L(X, \Delta) = \max(\{3\} \cup \{-(K_X + \Delta) \cdot B_\mu\})$$

where  $B_\mu$  ranges over the prime components of  $\Delta$  with  $B_\mu^2 < 0$ . Thus, the assertion follows from the following lemma.  $\square$

**Lemma 3.9.** *Let  $X$  be a normal surface and let  $\Delta$  be an  $\mathbb{R}$ -boundary. If  $C$  be a proper curve in  $X$  such that  $C^2 \leq 0$ , then  $-(K_X + \Delta) \cdot C \leq 2$ .*

*Proof.* Let  $f : Y \rightarrow X$  be the minimal resolution and let  $C_Y$  be the proper transform of  $C$ . We define  $\Delta_Y$  by

$$K_Y + C_Y + \Delta_Y = f^*(K_X + C).$$

Note that  $\Delta_Y \geq 0$  and  $C_Y \not\subset \text{Supp} \Delta_Y$ . Then, we see  $(K_Y + C_Y) \cdot C_Y \geq -2$ . We obtain

$$\begin{aligned} (K_X + \Delta) \cdot C &\geq (K_X + C) \cdot C \\ &= f^*(K_X + C) \cdot C_Y \\ &= (K_Y + C_Y + \Delta_Y) \cdot C_Y \\ &\geq (K_Y + C_Y) \cdot C_Y \\ &\geq -2. \end{aligned}$$

$\square$

## 4. CONE THEOREM

In this section we prove the cone theorem. We use the Bend-and-Break in the sense of Definition 3.6. Thus, in this section, we use the notations in Definition 3.6.

Here, let us recall the definition of the Kleiman–Mori cone.

**Definition 4.1.** Let  $X$  be a projective variety. Then we define

$$\begin{aligned} N(X) &:= \{r_1 Z_1 + \cdots + r_s Z_s \mid r_i \in \mathbb{R} \text{ and } Z_i \text{ is a curve in } X\} / \equiv \\ NE(X) &:= \{[r_1 Z_1 + \cdots + r_s Z_s] \mid r_i \geq 0, Z_i \text{ is a curve in } X\} \subset N(X) \end{aligned}$$

where  $[r_1 Z_1 + \cdots + r_s Z_s]$  means the equivalence class.

Note that  $N(X)$  is the quotient space by " $\equiv$ ". Although we often use the intersection theory by Mumford, we do NOT take the quotient by " $\equiv_{\text{Mum}}$ ". The numerical equivalence " $\equiv$ " is induced by the intersections only with  $\mathbb{R}$ -Cartier divisors.

In this section, we use the following lemma repeatedly.

**Lemma 4.2.** Let  $a, b \in \mathbb{R}$  and  $c, d \in \mathbb{R}_{>0}$ . Then,

$$\frac{a+b}{c+d} \leq \max\left\{\frac{a}{c}, \frac{b}{d}\right\}.$$

*Proof.* The proof is easy. Thus, we omit the proof.  $\square$

The following lemma is a key in this section.

**Lemma 4.3.** Let  $X$  be a projective normal surface and let  $\Delta$  be an effective  $\mathbb{R}$ -divisor such that  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier. Let  $\Delta = \sum b_i B_i$  be the prime decomposition. Let  $H$  be an  $\mathbb{R}$ -Cartier ample  $\mathbb{R}$ -divisor. If  $C$  is a curve in  $X$  such that  $C \cdot (K_X + \Delta) < 0$ , then there exists a curve  $E$  in  $X$  with the following properties.

- (1)  $E$  is rational or  $E = B_j$  for some  $j$  such that  $B_j^2 < 0$ .
- (2)  $0 < -E \cdot (K_X + \Delta) \leq L(X, \Delta)$ .
- (3)

$$\frac{-C \cdot (K_X + \Delta)}{C \cdot H} \leq \frac{-E \cdot (K_X + \Delta)}{E \cdot H}.$$

This proof is very similar to [Kollár–Mori, Theorem 1.13].

*Proof.* In this proof, we use the notation (BBI) and (BBII) in the sense of Definition 3.6. First, if  $C = B_j$  with  $B_j^2 < 0$ , then the assertion is obvious. We may assume that  $C \neq B_j$  for all  $B_j$  with  $B_j^2 < 0$ . Then, we can use (BBII). But, since we do not use  $c_0$ , we fix  $c_0 \in C \setminus \text{Sing} X$ .

Set  $C'_n := \alpha_n C_n$ . We consider the following number

$$\begin{aligned} M &:= \frac{-C \cdot (K_X + \Delta)}{C \cdot H} = \frac{-p^n C \cdot (K_X + \Delta)}{p^n C \cdot H} \\ &= \frac{-C'_n \cdot (K_X + \Delta) - Z_n \cdot (K_X + \Delta)}{C'_n \cdot H + Z_n \cdot H} = \frac{a_n + b_n}{c_n + d_n} \end{aligned}$$

where  $a_n$ ,  $b_n$ ,  $c_n$  and  $d_n$  are defined by

$$\begin{aligned} a_n &:= -C'_n \cdot (K_X + \Delta) \\ b_n &:= -Z_n \cdot (K_X + \Delta) \\ c_n &:= C'_n \cdot H \\ d_n &:= Z_n \cdot H. \end{aligned}$$

**Step 1.** In this step, we reduce the proof to the case where  $\alpha_n > 0$  for all  $n \gg 0$ .

Assume that there is a positive integer  $n$  such that  $n \geq n(X, \Delta, C, c_0)$  and  $\alpha_n = 0$ . Then we have

$$\frac{-C \cdot (K_X + \Delta)}{C \cdot H} = \frac{-Z_n \cdot (K_X + \Delta)}{Z_n \cdot H} \leq \frac{-Z_{n,i} \cdot (K_X + \Delta)}{Z_{n,i} \cdot H}$$

for some  $i$  by Lemma 4.2. Moreover, by (BBI) and Lemma 4.2, we obtain the desired result.

**Step 2.** In this step, we reduce the proof to the case where

$$a_n = -\alpha_n C_n \cdot (K_X + \Delta) \leq 2g(C_{\text{normal}})$$

for all  $n \gg 0$ .

Suppose the contrary. Then, by (c) of (BBII), we obtain  $C_n = B_j$  for some  $j$  such that  $B_j^2 < 0$ . By Lemma 4.2, we have the following equality

$$\begin{aligned} \frac{-C \cdot (K_X + \Delta)}{C \cdot H} &= \frac{-\alpha_n B_j \cdot (K_X + \Delta) - Z_n \cdot (K_X + \Delta)}{\alpha_n B_j \cdot H + Z_n \cdot H} \\ &\leq \max \left\{ \frac{-B_j \cdot (K_X + \Delta)}{B_j \cdot H}, \frac{-Z_n \cdot (K_X + \Delta)}{Z_n \cdot H} \right\}. \end{aligned}$$

If

$$\frac{-C \cdot (K_X + \Delta)}{C \cdot H} \leq \frac{-B_j \cdot (K_X + \Delta)}{B_j \cdot H},$$

then this is the desired result. If

$$\frac{-C \cdot (K_X + \Delta)}{C \cdot H} \leq \frac{-Z_n \cdot (K_X + \Delta)}{Z_n \cdot H},$$

then, by (BBI) and Lemma 4.2, we obtain the desired result.

From now on, we consider the asymptotic behaviors of  $a_n$ ,  $b_n$ ,  $c_n$  and  $d_n$ .

**Step 3.** The sequence  $a_n$  is bounded and  $b_n$  is not bounded.

Indeed, the boundedness of  $a_n$  follows from Step 2. Since  $a_n + b_n = -p^n C \cdot (K_X + \Delta)$  is not bounded,  $b_n$  is not bounded.

**Step 4.** In this step, we prove that for an arbitrary positive real number  $\epsilon$ , there exists a curve  $E$  in  $X$  with the following properties.

- (1)'  $E$  is rational.
- (2)'  $0 < -E \cdot (K_X + \Delta) \leq L(X, \Delta)$ .
- (3)'

$$M - \epsilon < \frac{-E \cdot (K_X + \Delta)}{E \cdot H}.$$

If  $a_n/c_n < M$  for some  $n \gg 0$ , then we have  $b_n/d_n \geq M$ , which gives us the desired result by (BBI) and Lemma 4.2. Thus, we may assume that  $a_n/c_n \geq M$  for all  $n \gg 0$ . Then, since  $a_n$  is bounded, so is  $c_n$  because  $M$  is a positive number. Because  $c_n + d_n = p^n C \cdot H$ ,  $d_n$  is not bounded. Therefore, for sufficiently large  $n$ , we obtain

$$\frac{b_n}{d_n} + \epsilon > \frac{a_n + b_n}{d_n} > \frac{a_n + b_n}{c_n + d_n} = M.$$

By (BBI) and Lemma 4.2, there exists a rational curve  $E$  with the desired properties.

**Step 5.** We take an arbitrary positive real number  $\epsilon$  with  $0 < \epsilon \leq M/2$ . Then, by Step 4, we obtain

$$E \cdot H < \frac{-E \cdot (K_X + \Delta)}{M - \epsilon} \leq \frac{L(X, \Delta)}{M/2} = \frac{2L(X, \Delta)}{M}.$$

Since  $H$  is ample, the following subset in numerical classes of effective 1-cycles in  $X$  with integral coefficients

$$\{[E] \mid E \cdot H < \frac{2L(X, \Delta)}{M}\}$$

has only finitely many members. Therefore, so is the following set

$$\left\{ \frac{-E \cdot (K_X + \Delta)}{E \cdot H} \mid E \cdot H < \frac{2L(X, \Delta)}{M} \text{ and } E \text{ satisfies (1)'(2)'} \right\}$$

because  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier. Take a sufficiently small  $\epsilon > 0$ . Then, by Step 4, we obtain a rational curve  $E$  in  $X$  such that

$$E \text{ satisfies (1)'(2)'} \text{ and } \frac{-E \cdot (K_X + \Delta)}{E \cdot H} \geq M.$$

This completes the proof. □

Let us prove the cone theorem.

**Theorem 4.4** (Cone theorem). *Let  $X$  be a projective normal surface and let  $\Delta$  be an effective  $\mathbb{R}$  divisor such that  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier. Let  $\Delta = \sum b_i B_i$  be the prime decomposition. Let  $H$  be an  $\mathbb{R}$ -Cartier ample  $\mathbb{R}$ -divisor. Then the following assertions hold:*

- (1)  $\overline{NE}(X) = \overline{NE}(X)_{K_X + \Delta \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$ .
- (2)  $\overline{NE}(X) = \overline{NE}(X)_{K_X + \Delta + H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i]$ .
- (3) Each  $C_i$  in (1) and (2) is rational or  $C_i = B_j$  for some  $B_j$  with  $B_j^2 < 0$ .
- (4) Each  $C_i$  in (1) and (2) satisfies  $0 < -C_i \cdot (K_X + \Delta) \leq L(X, \Delta)$ .

This proof is essentially the same as [Kollár-Mori, Theorem 1.24].

*Proof.* (1) Let  $W$  be the right hand side in (1), i.e.

$$W := \overline{NE}(X)_{(K_X + \Delta) \geq 0} + \sum_{C_i \text{ satisfies (3)(4)}} \mathbb{R}_{\geq 0}[C_i].$$

Note that  $W$  is a closed set by the same proof as in [Kollár-Mori, Theorem 1.24]. We would like to prove  $\overline{NE}(X) = W$ . The inclusion  $\overline{NE}(X) \supset W$  is clear. Let us assume  $\overline{NE}(X) \supsetneq W$  and derive a contradiction. Then we can find a Cartier divisor  $D$  which is positive on  $W \setminus 0$  and which is negative on some element of  $\overline{NE}(X)$ . Let  $\mu$  be a positive real number such that  $H + \mu D$  is nef and  $H + \mu' D$  is ample for all positive real numbers  $\mu'$  with  $\mu' < \mu$ . Then we can take a 1-cycle  $Z$  with  $Z \in \overline{NE}(X) \setminus \{0\}$  and  $(H + \mu D) \cdot Z = 0$ . Since  $Z \cdot H > 0$  means  $Z \cdot D < 0$ ,  $Z$  is not in  $W$ . By the definition of  $W$ , we obtain  $Z \cdot (K_X + \Delta) < 0$ . Because  $Z$  is an element of  $\overline{NE}(X)$ , there exist effective 1-cycles  $Z_k = \sum a_{k,j} Z_{k,j}$  such that the limit of  $Z_k$  is  $Z$ . Take an arbitrary positive real number  $\mu'$  with  $\mu' < \mu$ . By the ampleness of  $H + \mu' D$ , we have

$$\max_j \frac{-Z_{k,j} \cdot (K_X + \Delta)}{Z_{k,j} \cdot (H + \mu' D)} \geq \frac{-Z_k \cdot (K_X + \Delta)}{Z_k \cdot (H + \mu' D)}.$$

We may assume that the max on the left hand side occurs when  $j$  is zero. By Lemma 4.3, we obtain

$$\frac{-E_k \cdot (K_X + \Delta)}{E_k \cdot (H + \mu' D)} \geq \frac{-Z_{k,0} \cdot (K_X + \Delta)}{Z_{k,0} \cdot (H + \mu' D)} \geq \frac{-Z_k \cdot (K_X + \Delta)}{Z_k \cdot (H + \mu' D)}.$$

Here,  $E_k$  satisfies (3) and (4). Thus, we have  $E_k \in W$  and this means  $E_k \cdot D \geq 0$ . Therefore we have

$$\frac{-E_k \cdot (K_X + \Delta)}{E_k \cdot H} \geq \frac{-E_k \cdot (K_X + \Delta)}{E_k \cdot (H + \mu' D)} \geq \frac{-Z_k \cdot (K_X + \Delta)}{Z_k \cdot (H + \mu' D)}.$$

Take a large positive number  $r$  such that  $rH + (K_X + \Delta)$  is ample. This shows

$$r > \frac{-E_k \cdot (K_X + \Delta)}{E_k \cdot H}.$$

Combining the inequalities, we obtain

$$r > \frac{-Z_k \cdot (K_X + \Delta)}{Z_k \cdot (H + \mu' D)}.$$

Recall that we choose  $\mu'$  as an arbitrary positive real number with  $\mu' < \mu$ . By taking the limit  $\mu'$  to  $\mu$ , we obtain

$$r \geq \frac{-Z_k \cdot (K_X + \Delta)}{Z_k \cdot (H + \mu D)}.$$

Moreover, by taking the limit  $k$  to  $\infty$ , we obtain

$$r \geq \lim_{k \rightarrow \infty} \frac{-Z_k \cdot (K_X + \Delta)}{(Z_k \cdot H + \mu D)} = \frac{(\text{positive})}{+0} = +\infty.$$

This is a contradiction. This completes the proof of (1).

(2) If  $C_i \cdot (K_X + \Delta + H) < 0$ , then we have

$$C_i \cdot H < -C_i \cdot (K_X + \Delta) \leq L(X, \Delta).$$

There are only finitely many numerical classes of curves like this. This shows (2). The remaining assertions (3) and (4) have already proven in the above arguments.  $\square$

**Remark 4.5.** In Theorem 4.4,  $L(X, \Delta)$  gives an upper bound of length of extremal rays. By the proof of Theorem 3.7,

$$L(X, \Delta) := \max(\{3\} \cup \{-(K_X + \Delta) \cdot B_\mu\})$$

where  $B_\mu$  ranges over the prime components of  $\Delta$  with  $B_\mu^2 < 0$ . In the case where  $\Delta$  is an  $\mathbb{R}$ -boundary, we can set  $L(X, \Delta) = 3$  by Proposition 3.8.

Moreover, in the case where  $\Delta$  is an  $\mathbb{R}$ -boundary, every  $(K_X + \Delta)$ -negative extremal ray is generated by a rational curve.

**Proposition 4.6.** *Let  $X$  be a projective normal surface and let  $\Delta$  be an  $\mathbb{R}$ -boundary such that  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier. If  $R$  is a  $(K_X + \Delta)$ -negative extremal ray of  $\overline{NE}(X)$ , then  $R = \mathbb{R}_{\geq 0}[C]$  where  $C$  is a rational curve such that  $-(K_X + \Delta) \cdot C \leq 3$ .*

*Proof.* By Theorem 4.4 and Remark 4.5, we can write  $R = \mathbb{R}_{\geq 0}[C]$  where  $C$  is a curve such that  $-(K_X + \Delta) \cdot C \leq 3$  and that  $C$  is rational or  $C^2 < 0$ . Assume  $C^2 < 0$ . Then, we obtain

$$(K_X + C) \cdot C \leq (K_X + \Delta) \cdot C < 0.$$



Thus, the assertion follows from the following lemma.  $\square$

**Lemma 4.7.** *Let  $X$  be a normal surface and let  $C$  be a proper curve in  $X$ . If  $(K_X + C) \cdot C < 0$ , then  $C$  is a rational curve.*

*Proof.* Let  $f : Y \rightarrow X$  be the minimal resolution and let  $C_Y$  be the proper transform of  $C$ . We define  $\Delta_Y$  by  $K_Y + C_Y + \Delta_Y = f^*(K_X + C)$ . Then, we see

$$(K_Y + C_Y) \cdot C_Y \leq (K_Y + C_Y + \Delta_Y) \cdot C_Y = (K_X + C) \cdot C < 0.$$

Then, since  $C_Y$  is a rational curve, so is  $C$ .  $\square$

## 5. RESULTS ON ADJUNCTION FORMULA

In this section, we summarize results on adjunction formula.

**Proposition 5.1.** *Let  $X$  be a projective normal surface and let  $C$  be a curve in  $X$ . Then, there exists a exact sequence*

$$0 \rightarrow \mathcal{T} \rightarrow \omega_X(C)|_C \rightarrow \omega_C \rightarrow 0$$

where  $\mathcal{T}$  is the torsion subsheaf of  $\omega_X(C)|_C$ .

*Proof.* See [Fujino2, Lemma 4.4].  $\square$

Using this adjunction formula, we obtain the following result on global sections.

**Lemma 5.2.** *Let  $X$  be a projective normal surface and let  $C$  be a curve in  $X$  such that  $r(K_X + C)$  is Cartier for some positive integer  $r$ . If  $H^1(C, \mathcal{O}_C) \neq 0$ , then  $H^0(C, \omega_X(C)^{[r]}|_C) \neq 0$  where  $\omega_X(C)^{[r]}$  is the double dual of  $\omega_X(C)^{\otimes r}$ .*

*Proof.* We consider the exact sequence

$$0 \rightarrow \mathcal{T} \rightarrow \omega_X(C)|_C \rightarrow \omega_C \rightarrow 0.$$

Since  $\mathcal{T}$  is a skyscraper sheaf, we have  $H^1(C, \mathcal{T}) = 0$ . By  $H^0(C, \omega_C) \neq 0$ , we obtain  $H^0(C, \omega_X(C)|_C/\mathcal{T}) \neq 0$ . Thus there exists a map

$$\mathcal{O}_C \rightarrow \omega_X(C)|_C$$

such that this is injective on some non-empty open set. Therefore we obtain a map

$$\mathcal{O}_C \rightarrow \omega_X(C)^{\otimes r}|_C,$$

which is injective on some non-empty open set. On the other hand, there is a natural map

$$\omega_X(C)^{\otimes r}|_C \rightarrow \omega_X(C)^{[r]}|_C,$$

which is bijective on some non-empty open set. Combining these maps, we have the map

$$\mathcal{O}_C \rightarrow \omega_X(C)^{[r]}|_C,$$

which is injective on some non-empty open set. But this is actually injective because these two sheaves are invertible. This means  $H^0(C, \omega_X(C)^{[r]}|_C) \neq 0$ .  $\square$

Using this lemma, we obtain the following theorem, which plays a crucial role in this paper.

**Theorem 5.3.** *Let  $X$  be a projective normal surface and let  $C$  be a curve in  $X$  such that  $r(K_X + C)$  is Cartier for some positive integer  $r$ .*

- (1) *If  $C \cdot (K_X + C) < 0$ , then  $C \simeq \mathbb{P}^1$ .*
- (2) *If  $C \cdot (K_X + C) = 0$ , then  $C \simeq \mathbb{P}^1$  or  $\mathcal{O}_C((K_X + C)^{[r]}) \simeq \mathcal{O}_C$ .*

*Proof.* (1) Since  $C \cdot (K_X + C) < 0$  means  $H^0(C, \omega_X(C)^{[r]}|_C) = 0$ , the curve  $C$  must be  $\mathbb{P}^1$  by Lemma 5.2.

(2) Assume  $C \not\simeq \mathbb{P}^1$ . Then we can apply Lemma 5.2 and we obtain  $H^0(C, \omega_X(C)^{[r]}|_C) \neq 0$ . By  $C \cdot (K_X + C) = 0$ , we have  $\omega_X(C)^{[r]}|_C \simeq \mathcal{O}_C$ .  $\square$

## 6. CONTRACTION THEOREM

In this section, we show that extremal rays are contractable for  $\mathbb{Q}$ -factorial surfaces with  $\mathbb{R}$ -boundaries. First, we consider the following theorem, which we will use later.

**Theorem 6.1.** *In this theorem, let  $k$  be an algebraically closed field of arbitrary characteristic. Let  $\pi : Y \rightarrow B$  be a surjective morphism over  $k$  from a smooth projective surface  $Y$  to a smooth projective irrational curve  $B$ . Let  $f : Y \rightarrow X$  be a birational morphism to a projective normal  $\mathbb{Q}$ -factorial surface. If  $k$  is not the algebraic closure of a finite field, then all  $f$ -exceptional curves are  $\pi$ -vertical.*

This proof is essentially due to [Fujino2, Lemma 5.2].

*Proof.* We assume that  $C$  is an  $f$ -exceptional curve with  $\pi(C) = B$  and want to derive a contradiction. We may assume that  $C$  is smooth by taking a sequence of blow-ups of singular points of  $C$ . We have

$$\begin{aligned} \pi|_C : C &\longrightarrow B \\ \text{Pic}^0 C &\xleftarrow{(\pi|_C)^*} \text{Pic}^0 B. \end{aligned}$$

We prove that the image  $(\pi|_C)^*(\text{Pic}^0 B)$  is an abelian group whose rank is infinite. By considering  $(\pi|_C)^*$  as a morphism between Jacobian varieties, we see that  $(\pi|_C)^*(\text{Pic}^0 B)$  is an abelian variety. Note that the

dimension of  $(\pi|_C)^*(\text{Pic}^0 B)$  as a scheme is not zero by  $(\pi|_C)_* \circ (\pi|_C)^* = \deg(\pi|_C)$  and by the irrationality of  $B$ . Thus, by Fact 2.4, the rank of  $(\pi|_C)^*(\text{Pic}^0 B)$  is infinite. Then, we have

$$(\pi|_C)^*(\text{Pic}^0 B) \otimes_{\mathbb{Z}} \mathbb{Q} \setminus \sum_{i=1}^r \mathbb{Q}(E_i|_C) \neq \emptyset$$

where  $E_1, \dots, E_r$  are the  $f$ -exceptional curves. Therefore we can take a  $\mathbb{Q}$ -divisor  $D$  on  $B$  such that

$$(\pi^* D)|_C \notin \sum_{i=1}^r \mathbb{Q}(E_i|_C)$$

On the other hand, since  $X$  is  $\mathbb{Q}$ -factorial, we obtain

$$\pi^* D - f^* f_* \pi^* D \in \sum_{i=1}^r \mathbb{Q} E_i.$$

Restricting this relation to  $C$ , we have the following contradiction

$$(\pi^* D)|_C \in \sum_{i=1}^r \mathbb{Q}(E_i|_C)$$

because  $C$  is  $f$ -exceptional. □

Originally, [Fujino2] uses this theorem to prove the non-vanishing theorem. We use this theorem not only for the non-vanishing theorem but also for the following contraction theorem.

**Theorem 6.2** (Contraction theorem). *Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface and let  $\Delta$  be an  $\mathbb{R}$ -boundary. Let  $R = \mathbb{R}_{\geq 0}[C]$  be a  $(K_X + \Delta)$ -negative extremal ray. Then there exists a surjective morphism  $\phi_R : X \rightarrow Y$  to a projective variety  $Y$  with the following properties: (1)-(5).*

- (1) *Let  $C'$  be a curve on  $X$ . Then  $\phi_R(C')$  is one point iff  $[C'] \in R$ .*
- (2)  *$(\phi_R)_*(\mathcal{O}_X) = \mathcal{O}_Y$ .*
- (3) *If  $L$  is an invertible sheaf with  $L \cdot C = 0$ , then  $nL = (\phi_R)^* L_Y$  for some invertible sheaf  $L_Y$  on  $Y$  and for some positive integer  $n$ .*
- (4)  *$\rho(Y) = \rho(X) - 1$ .*
- (5)  *$Y$  is  $\mathbb{Q}$ -factorial if  $\dim Y = 2$ .*

We devide the proof into the three cases:  $C^2 > 0$ ,  $C^2 = 0$  and  $C^2 < 0$ .

*Proof of the case where  $C^2 > 0$ .*  $C^2 > 0$  shows that  $C$  is a nef and big divisor. Therefore for an arbitrary curve  $C'$ , there exists an effective Cartier divisor  $E$  and positive integers  $n$  and  $m$  such that  $nC \sim mC' + E$  by Kodaira's lemma. Since  $C$  generates an extremal ray, we have  $C' \equiv qC$  for some rational number  $q$ . Recall that we choose  $C'$  as an arbitrary curve. Thus we obtain  $\rho(X) = 1$  and  $-K_X$  is ample. Then let  $Y$  be one point and (1)(2)(4) are satisfied. We want to prove (3), that is, we must show that for a  $\mathbb{Q}$ -divisor  $D$  if  $D \equiv 0$ , then  $D$  is a torsion. It is sufficient to prove  $\kappa(X, D) \geq 0$ . Thus, we assume  $\kappa(X, D) = -\infty$  and derive a contradiction. Let  $f : X' \rightarrow X$  be the minimal resolution,  $D' = f^*D$  and  $K_{X'} + E' = f^*K_X$  where  $E'$  is an effective  $f$ -exceptional  $\mathbb{Q}$ -divisor. Then we obtain

$$\kappa(X', K_{X'}) \leq \kappa(X', K_{X'} + E') = \kappa(X, K_X) = -\infty.$$

First we prove that  $X'$  is an irrational ruled surface. By Serre duality, we obtain  $h^2(X', D') = h^0(X', K_{X'} - D')$ . Moreover we get

$$\begin{aligned} \kappa(X', K_{X'} - D') &\leq \kappa(X', K_{X'} + E' - D') \\ &= \kappa(X', f^*(K_X - D)) \\ &= \kappa(X, K_X - D) = -\infty \end{aligned}$$

by the anti-ampleness of  $K_X - D$ . Hence  $h^2(X', D') = 0$ . Then, by the Riemann–Roch theorem, we obtain

$$-h^1(X', D') = \chi(\mathcal{O}_{X'}) + \frac{1}{2}D' \cdot (D' - K_{X'}) = \chi(\mathcal{O}_{X'})$$

because  $D' = f^*D \equiv 0$ . This shows that

$$0 \geq -h^1(X', nD') = 1 - h^1(X', \mathcal{O}_{X'}).$$

Thus we obtain  $h^1(X', \mathcal{O}_{X'}) \geq 1$  and this means that  $X'$  is an irrational ruled surface. Let  $\pi : X' \rightarrow B$  be its ruling. Here, if  $k = \overline{\mathbb{F}}_p$ , then  $D$  is a torsion by Corollary 2.5. Hence we consider the case  $k \neq \overline{\mathbb{F}}_p$ . Then we can apply Theorem 6.1 and all  $f$ -exceptional curves are  $\pi$ -vertical. This shows that  $\pi$  factors through  $X' \rightarrow X \rightarrow B$ . A curve in a fiber has non-positive self-intersection number. But this is a contradiction because each curve in  $X$  is ample. This completes the proof of the case  $C^2 > 0$ .  $\square$

*Proof of the case where  $C^2 = 0$ .* First let us prove  $\rho(X) = 2$ . It is sufficient to show that for an arbitrary divisor  $F$ , if  $F \cdot C = F \cdot (K_X + \Delta) = 0$ , then  $F \equiv 0$ . We need the following lemma.

**Lemma 6.3.** *If  $D_1, D_2 \in C^\perp = \{D \mid D \text{ is a divisor and } D \cdot C = 0\}$ , then  $D_1 \cdot D_2 = 0$*

This proof is essentially due to [Mori2, Lemma 3.29].

*Proof of Lemma 6.3.* We consider the quadratic form  $Q : C_{\mathbb{R}}^{\perp} \rightarrow \mathbb{R}$ . Here we consider  $C_{\mathbb{R}}^{\perp}$  as a subvector-space of the numerical equivalence classes of  $\mathbb{R}$ -divisors and  $Q$  is defined by the self-intersection. We want to prove that  $Q$  is identically zero. Take a nef divisor  $G$  such that  $\overline{NE}(X) \cap G^{\perp} = \mathbb{R}_{\geq 0}[C]$ . By the nefness of  $G$ , we obtain  $G^2 \geq 0$ . But  $G^2$  must be 0 because  $G^2 > 0$  shows that  $G$  is nef and big. Then, by  $G \cdot C = 0$ , we obtain  $C^2 < 0$  and this is a contradiction. This shows that  $Q$  is zero in a non-empty dense subset of an open subset in  $C_{\mathbb{R}}^{\perp}$  by the cone theorem. Therefore  $Q$  must be identically zero.  $\square$

Since  $F \in C^{\perp}$ , we obtain  $D \cdot F = 0$  for any divisor  $D \in C^{\perp}$ .  $\mathbb{R}$ -subvector-space  $C_{\mathbb{R}}^{\perp}$  in numerical classes of divisors has codimension one. Take its basis  $D_1, \dots, D_{\rho-1}$ . Then we get the basis  $D_1, \dots, D_{\rho-1}, (K_X + \Delta)$  of the whole space. Indeed, by  $C \cdot (K_X + \Delta) \neq 0$ , these vectors are linearly independent. Since  $F \cdot D_1 = \dots = F \cdot D_{\rho-1} = F \cdot (K_X + \Delta) = 0$ , we get  $F \equiv 0$ . Thus, we obtain  $\rho(X) = 2$ .

Next, let us prove that the divisor  $C$  is semi-ample. By  $C^2 = 0$  and  $(K_X + \Delta) \cdot C < 0$ , we obtain  $K_X \cdot C < 0$ . Let  $f : X' \rightarrow X$  be a resolution. By

$$\begin{aligned} (f^*C)^2 &= f_*(f^*C) \cdot C = C \cdot C = 0 \quad \text{and} \\ K_{X'} \cdot f^*C &= f_*(K_{X'}) \cdot C = K_X \cdot C < 0, \end{aligned}$$

The Riemann–Roch theorem shows that  $\kappa(X', f^*C) \geq 1$ . Note that  $h^2(X', nf^*C) = h^0(X', K_{X'} - nf^*C) = 0$  for all  $n \gg 0$ . Therefore we get  $\kappa(X, C) = \kappa(X', f^*C) \geq 1$ .  $C^2 = 0$  implies  $\kappa(X, C) = 1$ . Then, by the following proposition,  $C$  is a semi-ample divisor.

**Proposition 6.4.** *Let  $X$  be a projective normal surface and  $L$  be a nef line bundle. If  $\kappa(X, L) = 1$ , then  $L$  is semi-ample.*

*Proof.* See [Fujita, Theorem 4.1].  $\square$

Hence the complete linear system  $|mC|$  induces a morphism  $\phi_R : X \rightarrow Y$  to a smooth projective curve  $Y$ . This morphism satisfies (1), (2) and (4). We would like to show (3). Take a line bundle  $L$  with  $L \cdot C = 0$ . Since  $\rho(X) = 2$ , we have  $L \equiv qC$  for some rational number  $q$ . We take a large positive integer  $s$  such that  $q + s$  is positive. Then we have

$$L + sC \equiv (q + s)C.$$

By the same argument as above, we see that  $L + sC$  is semi-ample. Then sufficiently large multiple of  $L + sC$  induces a morphism  $\psi : X \rightarrow Z$  to

a smooth projective curve  $Z$ . Moreover since this morphism satisfies the condition (1), we obtain the factorization

$$\psi : X \xrightarrow{\phi_R} Y \xrightarrow{\sigma} Z$$

with  $\sigma_*\mathcal{O}_Y = \mathcal{O}_Z$ . Since  $Y$  and  $Z$  are smooth projective curves,  $\sigma$  must be isomorphism. Then  $n(L + sC)$  is a pull-back of line bundle on  $Y$  for some positive integer  $n$ . This means (3).  $\square$

Before the proof of the case where  $C^2 < 0$ , we state a proposition on the contraction of  $\mathbb{P}^1$ .

**Proposition 6.5.** *Let  $X$  be a projective normal surface and let  $C$  be a curve in  $X$  isomorphic to  $\mathbb{P}^1$ . Assume  $G$  is a nef and big line bundle on  $X$  such that, for every curve  $C'$  in  $X$ ,  $G \cdot C' = 0$  iff  $C' = C$ . Then,  $G$  is semi-ample.*

*Proof.* By Keel's result (Theorem 2.2), if  $G|_C$  is semi-ample, then  $G$  is semi-ample. But, by  $C = \mathbb{P}^1$ , this is obvious.  $\square$

*Proof of the case where  $C^2 < 0$ .* By  $C \cdot (K_X + \Delta) < 0$ , we have  $C \cdot (K_X + C) < 0$ . Therefore, by Theorem 5.3, we see  $C \simeq \mathbb{P}^1$ . Let  $G$  be a nef and big divisor such that for any curve  $C'$ ,  $G \cdot C' = 0$  iff  $C' = C$ . (The way to construct such a divisor  $G$  is the following: let  $H$  be an ample divisor and  $G$  be the divisor such that  $G = H + qC$  and  $G \cdot C = 0$  for rational number  $q$ .) Then, by Proposition 6.5, there exists  $\phi_R$  satisfying (1) and (2). The remaining assertions (3), (4) and (5) hold from the following propositions.  $\square$

First we prove (3). We generalize the setting a little for a later use.

**Proposition 6.6** (Proof of (3)). *Let  $f : X \rightarrow Y$  be a proper birational morphism from a normal  $\mathbb{Q}$ -factorial surface  $X$  to a normal surface  $Y$ . Assume  $C := \text{Ex}(f)$  is a proper irreducible curve and  $f(C)$  is one point. Let  $L$  be a Cartier divisor on  $X$  with  $L \cdot C = 0$ . If  $L|_C$  is a torsion, then  $nL = f^*(L_Y)$  for some Cartier divisor  $L_Y$  on  $Y$  and for some positive integer  $n$ .*

*Proof.*

**Step 1.** In this step, we assume  $X$  and  $Y$  are projective and we prove the assertion.

Let  $G$  be the pull-back of an ample divisor. By Kodaira's lemma,  $G = A + E$  where  $A$  is an ample  $\mathbb{Q}$ -divisor and  $E$  is an effective  $\mathbb{Q}$ -divisor. By replacing  $G$  by its suitable multiple, it is easy to see that we may assume  $E = qC$  for some  $q \in \mathbb{Q}_{>0}$ . Consider the divisor

$$G' = mG + L = (mA + L) + mqC$$

for  $m \gg 0$ . Since  $mA + L$  is ample for  $m \gg 0$ , we see  $G' \cdot C' > 0$  for every curve  $C' \neq C$ . On the other hand, we have

$$G' \cdot C = (mG + L) \cdot C = 0.$$

Thus, for sufficiently large integer  $m \gg 0$ , the Cartier divisor  $G' = mG + L$  is nef and big such that  $G' \cdot C' = 0$  iff  $C' = C$  for every curve  $C'$ . Since  $L|_C$  is a torsion,  $G' = mG + L$  is semi-ample by Keel's result (Theorem 2.2). By Zariski's main theorem,  $|nG'|$  induces the same morphism as  $f$  for some  $n \in \mathbb{Z}_{>0}$ . Thus,  $nG' = nmG + nL$  is a pull-back of some line bundle on  $Y$ . So is the difference  $nL = nG' - nmG$ .

**Step 2.** In this step, we assume  $Y$  is quasi-projective and we prove the assertion.

Take a compactification  $Y \subset \bar{Y}$  such that  $\bar{Y}$  is projective and  $\bar{Y}$  is smooth on  $\bar{Y} \setminus Y$ . We define  $\bar{X}$  by patching  $X$  and  $\bar{Y}$  along  $X \setminus C \simeq Y \setminus \{f(C)\}$ . Then,  $\bar{X}$  is projective because  $\bar{X}$  is proper and  $\mathbb{Q}$ -factorial (cf. [Fujino2, Lemma 2.2]). Thus, by Step 1, we obtain the required assertion.

**Step 3.** In this step, we prove the assertion.

Let  $f(C) \in Y_0 \subset Y$  be an affine open subset and let  $X_0 := f^{-1}(Y_0)$ . Let  $f|_{X_0} =: f_0$ . Then, by Step 2, we obtain  $nL|_{X_0} = (f_0)^*L_{Y_0}$ . Let  $L_Y$  be the  $\mathbb{Z}$ -divisor on  $Y$  such that  $L_Y|_{Y_0} = L_{Y_0}$  and that  $L_Y$  has no prime component contained in  $Y \setminus Y_0$ . Then,  $L_Y$  is  $\mathbb{Q}$ -Cartier. Consider the following prime decomposition

$$L = \sum l_i C_i = \sum_{C_i \subset X \setminus X_0} l_i C_i + \sum_{C_j \not\subset X \setminus X_0} l_j C_j.$$

We see  $nf^*L_Y = \sum_{C_j \not\subset X \setminus X_0} l_j C_j$ . Since  $\sum_{C_j \subset X \setminus X_0} l_j C_j$  is the pull-back of some  $\mathbb{Q}$ -Cartier divisor, we obtain the assertion. □

The condition (4) is an immediate corollary from (3). Thus we would like to prove (5).

**Proposition 6.7** (Proof of (5)). *Let  $f : X \rightarrow Y$  be a proper birational morphism from a normal  $\mathbb{Q}$ -factorial surface  $X$  to a normal surface  $Y$ . Assume  $C := \text{Ex}(f)$  is a proper irreducible curve and  $f(C)$  is one point. Suppose the following condition.*

- (3) *If  $L$  is a Cartier divisor with  $L \cdot C = 0$ , then  $nL = (\phi_R)^*L_Y$  for some Cartier divisor  $L_Y$  on  $Y$  and for some positive integer  $n$ .*

*Then,  $Y$  is  $\mathbb{Q}$ -factorial.*

*Proof.* Let  $E$  be a prime divisor on  $Y$  and let  $D$  be its proper transform. Since  $C^2 < 0$ , there exists a rational number  $q$  such that  $(D + qC) \cdot C = 0$ . By (3), we have  $n(D + qC) = f^*(L_Y)$  for some Cartier divisor  $L_Y$ . By operating  $f_*$ , we obtain the equality  $nE = L_Y$  as Weil divisors. Therefore  $E$  is  $\mathbb{Q}$ -Cartier.  $\square$

Since we have the cone theorem and the contraction theorem, we obtain the minimal model program for  $\mathbb{Q}$ -factorial surfaces with boundaries.

**Theorem 6.8** (Minimal model program). *Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface and let  $\Delta$  be an  $\mathbb{R}$ -boundary. Then, there exists a sequence of projective birational morphisms*

$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\phi_0} (X_1, \Delta_1) \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{s-1}} (X_s, \Delta_s) =: (X^\dagger, \Delta^\dagger)$$

where  $(\phi_{i-1})_*(\Delta_{i-1}) =: \Delta_i$

with the following properties.

- (1) Each  $X_i$  is a projective normal  $\mathbb{Q}$ -factorial surface.
- (2) Each  $\Delta_i$  is an  $\mathbb{R}$ -boundary.
- (3) For each  $i$ ,  $\text{Ex}(\phi_i) =: C_i$  is an irreducible curve such that such that

$$(K_{X_i} + \Delta_i) \cdot C_i < 0.$$

- (4)  $(X^\dagger, \Delta^\dagger)$  satisfies one of the following conditions.
  - (a)  $K_{X^\dagger} + \Delta^\dagger$  is nef.
  - (b) There is a projective surjective morphism  $\mu : X^\dagger \rightarrow Z$  to a smooth projective curve  $Z$  such that  $\mu_* \mathcal{O}_{X^\dagger} = \mathcal{O}_Z$ ,  $-(K_{X^\dagger} + \Delta^\dagger)$  is  $\mu$ -ample and  $\rho(X^\dagger) = 2$ .
  - (c)  $-(K_{X^\dagger} + \Delta^\dagger)$  is ample and  $\rho(X^\dagger) = 1$ .

## 7. FINITE GENERATION OF CANONICAL RINGS

It is important to consider the finite generation of canonical rings, which is closely related to the minimal model program. In this section, we prove the following theorem.

**Theorem 7.1** (Finite generation theorem). *Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface over  $k$  and let  $\Delta$  be a  $\mathbb{Q}$ -boundary. Then  $R(X, K_X + \Delta) := \bigoplus_{m \geq 0} H^0(X, \lfloor m(K_X + \Delta) \rfloor)$  is a finitely generated  $k$ -algebra.*

*Proof.* Let us consider the Kodaira dimension  $\kappa := \kappa(X, K_X + \Delta)$ . It is obvious for the case  $\kappa = -\infty$  and the case  $\kappa = 0$ . In particular we may assume that  $K_X + \Delta$  is effective. Then, by Theorem 6.8, we may



assume that  $K_X + \Delta$  is nef. The case  $\kappa = 1$  follows from Proposition 6.4. Therefore we may assume  $\kappa = 2$ , that is,  $K_X + \Delta$  is nef and big. This case follows from Proposition 7.2.  $\square$

**Proposition 7.2.** *Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface and let  $\Delta$  be a  $\mathbb{Q}$ -boundary. If  $K_X + \Delta$  is nef and big, then  $K_X + \Delta$  is semi-ample.*

*Proof.* By Keel's result (Theorem 2.2), it is sufficient to prove that if

$$E := \bigcup_{C \cdot (K_X + \Delta) = 0} C = C_1 \cup \cdots \cup C_r,$$

then  $(K_X + \Delta)|_E$  is semi-ample. Let  $C \subset E$ . Then we have

$$(K_X + C) \cdot C \leq (K_X + \Delta) \cdot C = 0.$$

**Step 1.** In this step, we reduce the proof to the case where if  $C \subset E$ , then  $(K_X + C) \cdot C = 0$ .

Assume  $C \subset E$  and  $(K_X + C) \cdot C < 0$ . Then  $C$  is a  $(K_X + C)$ -negative extremal curve. Thus, by Theorem 6.2, we can contract  $C$ . Let  $f : X \rightarrow Y$  be its contraction and  $\Delta_Y := f_*(\Delta)$ . Then since  $K_X + \Delta = f^*(K_Y + \Delta_Y)$  and  $Y$  is  $\mathbb{Q}$ -factorial, if we can prove that  $K_Y + \Delta_Y$  is semi-ample, then  $K_X + \Delta$  is semi-ample. We can repeat this procedure and we obtain the desired reduction.

**Step 2.** In this step, we prove that  $E$  is a disjoint union of irreducible curves and if  $C \subset E$ , then  $(K_X + \Delta)|_C = (K_X + C)|_C$ .

Let  $C \subset E$ . By Step 1, we have  $(K_X + C) \cdot C = 0$ . Then, the inequality over Step 1 is an equality. Thus  $C \subset \text{Supp} \Delta$  and  $C$  is disjoint from any other component of  $\Delta$ .

By Step 2, it is sufficient to prove that, if  $(K_X + C) \cdot C = 0$ , then  $(K_X + C)|_C$  is semi-ample. This is satisfied by Theorem 5.3.  $\square$

## 8. ABUNDANCE THEOREM ( $k \neq \overline{\mathbb{F}}_p$ )

In this section, we prove the abundance theorem for  $\mathbb{Q}$ -factorial surfaces with  $\mathbb{Q}$ -boundary over  $k \neq \overline{\mathbb{F}}_p$ . The case where  $k = \overline{\mathbb{F}}_p$  will be treated in Part 3. In the case where  $\kappa(X, K_X + \Delta) = 0$ , we give the proof of abundance theorem which does not depend on the characteristic of the base field  $k$  (Theorem 8.5).

First we see the following non-vanishing theorem.

**Theorem 8.1** (Non-vanishing theorem). *In this theorem, let  $k$  be an algebraically closed field of arbitrary characteristic. Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface over  $k$  and let  $\Delta$  be a  $\mathbb{Q}$ -boundary. If*

$k$  is not the algebraic closure of a finite field and  $K_X + \Delta$  is pseudo-effective, then  $\kappa(X, K_X + \Delta) \geq 0$ .

*Proof.* See [Fujino2, Theorem 5.1, Lemma 5.2]. Note that, instead of Lemma 5.2 of [Fujino2], we use Theorem 6.1 of this paper.  $\square$

Before the proof of the abundance theorem, we see the definition of indecomposable curves of canonical type in the sense of [Mumford2, P.330].

**Definition 8.2.** In this definition, let  $k$  be an algebraically closed field of arbitrary characteristic. Let  $X$  be a smooth projective surface over  $k$  and let  $Y = \sum n_i E_i$  be an effective divisor with  $n_i \in \mathbb{Z}_{>0}$ . We say  $Y$  is an *indecomposable curve of canonical type* if  $K_X \cdot E_i = Y \cdot E_i = 0$  for all  $i$ ,  $\text{Supp} Y$  is connected and  $\gcd(n_i) = 1$ .

We see criteria for the movability of indecomposable curves of canonical type.

**Proposition 8.3.** *In this proposition, let  $k$  be an algebraically closed field of arbitrary characteristic. Let  $X$  be a smooth projective surface over  $k$  and let  $Y$  be an indecomposable curve of canonical type in  $X$ . Assume that one of the following assertions holds:*

- (1)  $\text{char } k = p > 0$ .
- (2)  $H^1(X, \mathcal{O}_X) = 0$ .

*If  $\mathcal{O}_Y(Y)$  is a torsion, then  $\kappa(X, Y) = 1$ .*

The proof of (2) is very similar to [Totaro, Theorem 2.1].

*Proof.* (1) See [Mašek, Lemma in P.682].

(2) Assume  $H^1(X, \mathcal{O}_X) = 0$ . Let  $m$  be the order of  $\mathcal{O}_Y(Y)$  and  $a$  be an integer with  $1 \leq a \leq m - 1$ . The case where  $\mathcal{O}_Y(Y) = \mathcal{O}_Y$  is easy. So we exclude this case and we can assume  $m \geq 2$ . Let us consider the following exact sequence:

$$H^0(Y, \mathcal{O}_Y(aY)) \rightarrow H^1(X, (a-1)Y) \rightarrow H^1(X, aY) \rightarrow H^1(Y, \mathcal{O}_Y(aY)).$$

By Serre duality and [Mumford2, Corollary 1 in P.333], we obtain

$$h^1(Y, \mathcal{O}_Y(aY)) = h^0(Y, \mathcal{O}_Y(K_Y - aY)) = h^0(Y, \mathcal{O}_Y(-aY)).$$

The choice of  $a$  shows that

$$h^0(Y, \mathcal{O}_Y(aY)) = h^0(Y, \mathcal{O}_Y(-aY)) = 0.$$

Indeed, suppose the contrary, that is, for example suppose  $h^0(Y, aY|_Y) \neq 0$ . Then we have  $\mathcal{O}_Y(aY) = \mathcal{O}_Y$  by [Mumford2, Lemma in P.332]. This is a contradiction. Therefore we get

$$0 = h^1(X, \mathcal{O}_X) = h^1(X, Y) = \cdots = h^1(X, (m-1)Y).$$

This leads the following exact sequence:

$$0 \rightarrow H^0(X, (m-1)Y) \rightarrow H^0(X, mY) \rightarrow H^0(Y, \mathcal{O}_Y) \rightarrow 0.$$

Thus  $Y$  is an effective semi-ample divisor on  $X$  and  $Y^2 = 0$ . This shows that  $\kappa(X, Y) = 1$ .  $\square$

Now, we prove the abundance theorem.

**Theorem 8.4** (Abundance theorem). *In this theorem, let  $k$  be an algebraically closed field of positive characteristic. Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface over  $k$  and let  $\Delta$  be a  $\mathbb{Q}$ -boundary. If  $k$  is not the algebraic closure of a finite field and  $K_X + \Delta$  is nef, then  $K_X + \Delta$  is semi-ample.*

*Proof.* By Theorem 8.1, we may assume  $\kappa(K_X + \Delta) \geq 0$ . Moreover, we may assume  $\kappa(K_X + \Delta) = 0$  by Proposition 6.4 and Proposition 7.2. Thus it is sufficient to prove the following theorem.  $\square$

**Theorem 8.5.** *In this theorem, let  $k$  be an algebraically closed field of arbitrary characteristic. Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface over  $k$  and let  $\Delta$  be a  $\mathbb{Q}$ -boundary. If  $k$  is not the algebraic closure of a finite field and  $\kappa(X, K_X + \Delta) = 0$ , then  $K_X + \Delta \sim_{\mathbb{Q}} 0$ .*

This proof is very similar to [Fujino2, Theorem 6.1] and uses many techniques in [Fujita, section 5].

*Proof.* Let  $f : V \rightarrow X$  be the minimal resolution. We set  $K_V + \Delta_V = f^*(K_X + \Delta)$ . We note that  $\Delta_V$  is effective. It is sufficient to see that  $K_V + \Delta_V \sim_{\mathbb{Q}} 0$ . Let

$$\varphi : V =: V_0 \xrightarrow{\varphi_0} V_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{k-1}} V_k =: S$$

be a sequence of blow-downs such that

- (1)  $\varphi_i$  is a blow-down of a  $(-1)$ -curve  $C_i$  on  $V_i$ ,
- (2)  $\Delta_{V_{i+1}} = \varphi_{i*} \Delta_{V_i}$ , and
- (3)  $(K_{V_i} + \Delta_{V_i}) \cdot C_i = 0$ ,

for every  $i$ . We can assume that there are no  $(-1)$ -curves  $C$  on  $S$  with  $(K_S + \Delta_S) \cdot C = 0$ . We note that  $K_V + \Delta_V = \varphi^*(K_S + \Delta_S)$ . It is sufficient to see that  $K_S + \Delta_S \sim_{\mathbb{Q}} 0$ . Since  $\kappa(S, K_S + \Delta_S) = 0$ , there is a member  $Z$  of  $|m(K_S + \Delta_S)|$  for some divisible positive integer  $m$ . Then, for every positive integer  $t$ ,  $tZ$  is the unique member of  $|tm(K_S + \Delta_S)|$ . We will derive a contradiction assuming  $Z \neq 0$ .

**Step 1.** In this step, we prove that for each prime component  $Z_i$  of  $Z$ , we have

$$K_S \cdot Z_i = \Delta_S \cdot Z_i = Z \cdot Z_i = 0.$$

Since  $(K_S + \Delta_S) \cdot Z = m(K_S + \Delta_S)^2 = 0$  and  $(K_S + \Delta_S)$  is nef,  $(K_S + \Delta_S) \cdot Z_i = 0$  for all  $i$ . This means

$$Z \cdot Z_i = 0$$

and  $Z_i^2 \leq 0$ . Now, we prove  $K_S \cdot Z_i \geq 0$  for every  $i$ . If  $K_S \cdot Z_i < 0$ , then we obtain  $(K_S + Z_i) \cdot Z_i < 0$  and  $Z_i \cong \mathbb{P}^1$ . If  $Z_i^2 \geq 0$ , then we obtain  $\kappa(S, Z) \geq \kappa(S, Z_i) > 0$ . This contradicts  $\kappa(S, K_S + \Delta_S) = \kappa(S, Z) = 0$ . If  $Z_i^2 < 0$ , then  $Z_i$  is a  $(-1)$ -curve with  $(K_S + \Delta_S) \cdot Z_i = 0$ . This contradicts the definition of  $S$ . Anyway, we have  $K_S \cdot Z_i \geq 0$  for every  $i$ . This implies  $K_S \cdot Z = K_S \cdot m(K_S + \Delta_S) \geq 0$ . The nefness of  $K_S + \Delta_S$  shows  $(K_S + \Delta_S) \cdot \Delta_S \geq 0$ . By  $(K_S + \Delta_S)^2 = 0$ , we see  $(K_S + \Delta_S) \cdot K_S = (K_S + \Delta_S) \cdot \Delta_S = 0$ . This is equivalent to  $Z \cdot K_S = Z \cdot \Delta_S = 0$ . Since  $K_S \cdot Z_i \geq 0$ , we see

$$K_S \cdot Z_i = \Delta_S \cdot Z_i = 0.$$

**Step 2.** We can decompose  $Z$  into the connected components as follows:

$$Z = \sum_{i=1}^r \mu_i Y_i,$$

where  $\mu_i Y_i$  is a connected component of  $Z$  such that  $\mu_i$  is the greatest common divisor of the coefficients of prime components of  $Y_i$  in  $Z$  for every  $i$ . Then we see that, for every  $i$ , each  $Y_i$  is an indecomposable curve of canonical type by Step 1. We obtain  $\omega_{Y_i} \simeq \mathcal{O}_{Y_i}$  by [Mumford2, Corollary 1 in P.333].

**Step 3.** In this step, we assume  $\kappa(S, K_S) \geq 0$  and prove the assertion. Since

$$0 \leq \kappa(S, K_S) \leq \kappa(S, K_S + \Delta_S) = 0,$$

we obtain  $\kappa(S, K_S) = 0$ . Let us prove  $S$  is a minimal surface.

Suppose the contrary, that is, suppose that there exists a  $(-1)$ -curve  $E$ . Then we have the contraction  $g : S \rightarrow S'$  of  $E$  and we obtain a morphism

$$S \xrightarrow{g} S' \xrightarrow{h} S_{\min}$$

to a minimal surface  $S_{\min}$ . Since  $0 = \kappa(S, K_S) = \kappa(S_{\min}, K_{S_{\min}})$ , we see  $K_{S_{\min}} \sim_{\mathbb{Q}} 0$ . Because

$$K_S = g^* K_{S'} + E = g^*(h^*(K_{S_{\min}}) + (\text{effective divisor})) + E,$$

we see that  $K_S \sim_{\mathbb{Q}} (\text{effective divisor}) + E$ . This means that

$$nZ \sim nm(K_S + \Delta_S) \sim (\text{effective divisor}) + nmE + nm\Delta_S$$

for some  $n \in \mathbb{Z}_{>0}$ . Since  $\kappa(S, Z) = 0$ ,

$$nZ = (\text{effective divisor}) + nmE + nm\Delta_S$$

as Weil divisors. In particular, we have  $E \subset \text{Supp } Z$  and  $E = Z_i$  for some  $i$ . This implies that  $Z_i \cdot (K_S + \Delta_S) = 0$  and  $Z_i$  is a  $(-1)$ -curve and this is a contradiction to the construction of  $S$ . Therefore,  $S$  is minimal.

Then we obtain the following contradiction

$$\kappa(S, K_S + \Delta_S) = \kappa(S, Z) \geq \kappa(S, Y_i) \geq 1$$

from the known result  $\kappa(S, Y_i) \geq 1$ . (See, for example, [Bădescu, Theorem 7.11]. )

**Step 4.** By Step 3, we may assume that  $\kappa(S, K_S) = -\infty$ . In Step 5 and Step 6, we assume that  $S$  is rational and prove the assertion. In Step 7–Step 12 we assume that  $S$  is irrational and prove the assertion. Note that since  $\kappa(X, Z) = 0$ , in order to derive a contradiction, we want to prove that  $\kappa(X, Y_i) \geq 1$  for some  $i$ .

We assume that  $S$  is rational.

**Step 5.** In this step, we prove

$$\Delta_S = \sum y_i Y_i \text{ and } y_i > 1.$$

We fix  $i$ . By  $H^1(S, \mathcal{O}_S(K_S)) = 0$  and the following exact sequence

$$0 \rightarrow \mathcal{O}_S(K_S) \rightarrow \mathcal{O}_S(K_S + Y_i) \rightarrow \omega_{Y_i} \rightarrow 0,$$

we obtain the surjection

$$H^0(S, \mathcal{O}_S(K_S + Y_i)) \rightarrow H^0(Y_i, \omega_{Y_i}) \simeq H^0(Y_i, \mathcal{O}_{Y_i}).$$

Thus there exists  $W_i \in |K_S + Y_i|$  such that  $W_i$  has no components of  $Y_i$ . For  $\tilde{Z}_i = Z - \mu_i Y_i$ , we obtain the equation

$$m\mu_i W_i + m\mu_i \Delta_S + m\tilde{Z}_i = (\mu_i + m)Z.$$

Note that this equality holds as Weil divisors because  $\kappa(S, Z) = 0$ . From this equation,  $\text{Supp } \Delta_S \subset \text{Supp } Z$ . Since  $W_i$  and  $\tilde{Z}_i$  are free from the components of  $Y_i$ , we have  $\Delta_S = \sum \frac{\mu_i + m}{m} Y_i$ . We set  $y_i := \frac{\mu_i + m}{m} > 1$ .

We fix  $i$ , and we denote  $Y$  instead of  $Y_i$ .

**Step 6.** In this step, we prove the desired assertion. By (2) in Proposition 8.3, it is sufficient to prove that  $\mathcal{O}_Y(aY) \simeq \mathcal{O}_Y$  for some positive integer  $a$ . We set  $Y_{(k)} := Y$  and construct  $Y_{(j)}$  inductively. It is easy to see that  $\varphi_j : V_j \rightarrow V_{j+1}$  is the blow-up at  $P_{j+1}$  with  $\text{mult}_{P_{j+1}} \Delta_{V_{j+1}} \geq 1$  for every  $j$  since  $\Delta_{V_j}$  is effective. If  $\text{mult}_{P_{j+1}} Y_{(j+1)} = 0$ , then we set  $Y_{(j)} = \varphi_j^* Y_{(j+1)}$ . If  $\text{mult}_{P_{j+1}} Y_{(j+1)} > 0$ , then we set  $Y_{(j)} = \varphi_j^* Y_{(j+1)} - C_j$ , where  $C_j$  is the exceptional curve of  $\varphi_j$ . Thus, we obtain  $Y_{(0)}$  on  $V_0 = V$ . Note that  $\text{mult}_P \Delta_{V_{j+1}} > \text{mult}_P Y_{(j+1)}$  for every  $P \in \text{Supp } Y_{(j+1)}$  by Step 5 and

the above inductive construction. Moreover, since  $\text{mult}_P Y_{(j+1)} \in \mathbb{Z}$ , we see that  $Y_{(0)}$  is effective and that  $\text{Supp} Y_{(0)} \subset \text{Supp} \Delta_V^{>1}$  where, for the prime decomposition  $\Delta_V = \sum \delta_l \Delta_{V,l}$ , we define  $\Delta_V^{>1} := \sum_{\delta_l > 1} \delta_l \Delta_{V,l}$ . Then, we have  $\varphi_{j*} \mathcal{O}_{Y_{(j)}} \simeq \mathcal{O}_{Y_{(j+1)}}$  for every  $j$ . Indeed,  $\varphi_{j*} \mathcal{O}_{V_j}(-Y_{(j)}) \simeq \mathcal{O}_{V_{j+1}}(-Y_{(j+1)})$  and  $R^1 \varphi_{j*} \mathcal{O}_{V_j}(-Y_{(j)}) = 0$  for every  $j$ . See the following commutative diagram.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_{V_{j+1}}(-Y_{(j+1)}) & \longrightarrow & \mathcal{O}_{V_{j+1}} & \longrightarrow & \mathcal{O}_{Y_{(j+1)}} \longrightarrow 0 \\
& & \downarrow \simeq & & \downarrow \simeq & & \downarrow \\
0 & \longrightarrow & \varphi_{j*} \mathcal{O}_{V_j}(-Y_{(j)}) & \longrightarrow & \varphi_{j*} \mathcal{O}_{V_j} & \longrightarrow & \varphi_{j*} \mathcal{O}_{Y_{(j)}} \longrightarrow 0
\end{array}$$

Therefore, we obtain  $\varphi_* \mathcal{O}_{Y_{(0)}} \simeq \mathcal{O}_Y$ . Since  $\text{Supp} Y_{(0)} \subset \text{Supp} \Delta_V^{>1}$ , we see that  $Y_{(0)}$  is  $f$ -exceptional. Since  $K_V + \Delta_V = f^*(K_X + \Delta)$ , we obtain  $\mathcal{O}_{Y_{(0)}}(b(K_V + \Delta_V)) \simeq \mathcal{O}_{Y_{(0)}}$  for some positive divisible integer  $b$ . Thus,

$$\mathcal{O}_Y(b(K_S + \Delta_S)) \simeq \varphi_* \mathcal{O}_{Y_{(0)}}(b(K_V + \Delta_V)) \simeq \varphi_* \mathcal{O}_{Y_{(0)}} \simeq \mathcal{O}_Y.$$

This means

$$\mathcal{O}_Y(\mu b Y) \simeq \mathcal{O}_Y(b Z) \simeq \mathcal{O}_Y(b m(K_S + \Delta_S)) \simeq \mathcal{O}_Y.$$

This completes the proof of the rational case.

We assume that  $S$  is an irrational ruled surface. Let  $\pi : S \rightarrow B$  be its ruling and let  $F$  be one of its smooth fibers.

**Step 7.** In this step, we prove that each connected component  $\mu Y$  of  $Z$  satisfies  $F \cdot Y > 0$ .

We assume that  $F \cdot Y = 0$  and derive a contradiction. Since  $Y$  is connected,  $Y$  is contained in some fiber  $F_0$ . Then we have the equality

$$F_0 = yY + Y'$$

for some effective  $\mathbb{Q}$ -divisor  $Y'$  with  $\text{Supp} Y \not\subset \text{Supp} Y'$  and for some positive rational number  $y$ . By  $K_S \cdot F_0 = -2$  and  $K_S \cdot Y = 0$ , it is sufficient to prove  $Y' = 0$ . Thus assume  $Y' \neq 0$ . Take a prime component  $Y_{(1)}$  of  $Y$  which is not a component of  $Y'$ . The equalities  $F_0 \cdot Y_{(1)} = Y \cdot Y_{(1)} = 0$  show  $Y' \cdot Y_{(1)} = 0$ . Thus if  $Y_{(2)}$  is a prime component of  $Y$  such that  $Y_{(1)} \cap Y_{(2)} \neq \emptyset$ , then  $Y_{(2)}$  is not a component of  $Y'$ . By repeating this procedure, we see that  $Y_{(i)}$  is not a prime component of  $Y'$  for each prime component  $Y_{(i)}$  of  $Y$ . Since  $Y' \neq 0$ , there exists a prime component  $Y_{(j)}$  of  $Y$  with  $Y_{(j)} \cap Y' \neq \emptyset$ . This leads to the following contradiction

$$F_0 \cdot Y_{(j)} = Y \cdot Y_{(j)} = 0 \text{ and } Y' \cdot Y_{(j)} \neq 0.$$

**Step 8.** In this step, we prove that both  $B$  and  $Y$  are elliptic curves. In particular,  $Y^2 = 0$ .

By Step 7,  $Y$  has a prime component  $Y_{(0)}$  with  $\pi(Y_{(0)}) = B$ . Because  $(K_S + Y_{(0)}) \cdot Y_{(0)} \leq 0$  by Step 1,  $Y_{(0)}$  is a rational curve or an elliptic curve. But, since  $B$  is irrational and  $\pi(Y_{(0)}) = B$ ,  $Y_{(0)}$  must be an elliptic curve. Then, so is  $B$ . Moreover if an indecomposable curve of canonical type  $Y$  is reducible, then every prime component of  $Y$  must be  $\mathbb{P}^1$ .

Indeed, for every prime component  $Y_{(i)}$ , we have  $(K_S + Y_{(i)}) \cdot Y_{(i)} \leq 0$ . Assume that a prime component  $Y_{(0)}$  satisfies  $(K_S + Y_{(0)}) \cdot Y_{(0)} = 0$ . Then, by  $K_S \cdot Y_{(0)} = 0$ , we have  $Y_{(0)}^2 = 0$ . Since  $Y \cdot Y_{(0)} = 0$ , we obtain  $Y_{(i)} \cdot Y_{(0)} = 0$  for every prime component  $Y_{(i)}$ . Because  $Y$  is connected,  $Y$  must be irreducible.

**Step 9.** In this step, we prove that the coefficient  $\delta$  of  $Y$  in  $\Delta_S$  satisfies  $0 \leq \delta \leq 1$ .

Assume the contrary, that is, we assume  $\delta > 1$  and derive a contradiction. Take the proper transform  $Y_V$  of  $Y$  in  $V$ . We see that the coefficient of  $Y_V$  in  $\Delta_V$  is  $\delta$ . Then  $Y_V$  is contracted by  $f$  because the assumption of the boundary. Since  $X$  is  $\mathbb{Q}$ -factorial, we can apply Theorem 6.1 and this is a contradiction.

**Step 10.** In this step, we prove that  $\mathcal{O}_Y(Y)$  is a torsion.

By Step 1, we have  $Y \cdot (\Delta_S - \delta Y) = 0$ . This means  $\text{Supp} Y \cap \text{Supp}(\Delta_S - \delta Y) = \emptyset$ . Thus, in  $\text{Pic } Y$ , we obtain

$$\mu Y = Z = m(K_S + \Delta_S) = m(K_S + \delta Y) = m(-Y + \delta Y).$$

Therefore we have  $(m(1 - \delta) + \mu)Y = 0$  in  $\text{Pic } Y$ . By  $m(1 - \delta) + \mu > m(1 - \delta) \geq 0$ ,  $\mathcal{O}_Y(Y)$  must be a torsion.

**Step 11.** Let  $r$  be the order of the torsion  $\mathcal{O}_Y(Y)$ . In this step, we prove that

$$H^1(S, \mathcal{O}_S(K_S + tY)) = 0$$

for  $1 \leq t \leq r$  by induction.

Let us consider the exact sequence

$$0 \rightarrow \mathcal{O}_S(K_S) \rightarrow \mathcal{O}_S(K_S + Y) \rightarrow \omega_Y \rightarrow 0.$$

If the induced map

$$H^0(S, \mathcal{O}_S(K_S + Y)) \rightarrow H^0(Y, \omega_Y) = k$$

is surjective, we get a contradiction by the same argument as Step 5. Therefore, this map is zero. Then, the injective map

$$k = H^0(Y, \omega_Y) \rightarrow H^1(S, \mathcal{O}_S(K_S)) = k$$

is bijective. This means that the map

$$H^1(S, \mathcal{O}_S(K_S + Y)) \rightarrow H^1(Y, \omega_Y)$$

is injective. On the other hand, we have  $h^2(S, \mathcal{O}_S(K_S + Y)) = 0$  by Serre duality. Then we obtain the surjective map

$$H^1(Y, \omega_Y) \rightarrow H^2(S, \mathcal{O}_S(K_S)).$$

But this is bijective by Serre duality. Therefore we obtain

$$H^1(S, \mathcal{O}_S(K_S + Y)) = 0$$

and this proves the case where  $t = 1$ . When  $1 < t \leq r$ , we have the exact sequence:

$$H^1(S, K_S + (t-1)Y) \rightarrow H^1(S, K_S + tY) \rightarrow H^1(Y, K_S + tY).$$

By the induction hypothesis, we have  $H^1(S, K_S + (t-1)Y) = 0$ . Moreover, we obtain  $h^1(Y, \mathcal{O}_Y(K_S + tY)) = h^0(Y, \mathcal{O}_Y(-(t-1)Y)) = 0$  since  $r$  is the order of  $\mathcal{O}_Y(Y)$ . Thus we see  $H^1(S, K_S + tY) = 0$  for  $1 \leq t \leq r$ .

**Step 12.** By Step 11, we obtain a surjection

$$H^0(S, \mathcal{O}_S(K_S + (r+1)Y)) \rightarrow H^0(Y, \mathcal{O}_Y(K_S + (r+1)Y)).$$

By  $\mathcal{O}_Y(K_S + (r+1)Y) = \mathcal{O}_Y(K_Y + rY) = \mathcal{O}_Y$ , there exists an effective member  $W \in |K_S + (r+1)Y|$  free from  $Y$ . Set  $\tilde{Z} := Z - \mu Y$ . We obtain the equation

$$\begin{aligned} \mu Z + (r+1)mZ &= \mu m(K_S + \Delta_S) + (r+1)m(\mu Y + \tilde{Z}) \\ &= \mu m(K_S + (r+1)Y) + \mu m\Delta_S + (r+1)m\tilde{Z} \\ &= \mu mW + \mu m\Delta_S + (r+1)m\tilde{Z} \end{aligned}$$

as Weil divisors. By considering the coefficients of  $Y$  in both sides, we obtain

$$(\mu + (r+1)m)\mu = \mu m\delta.$$

But these two numbers are different by  $0 \leq \delta \leq 1$ . This is a contradiction.  $\square$

**Remark 8.6.** If  $\text{char } k > 0$ , then we do not need Step 11 and Step 12 in the proof of Theorem 8.5 by using (1) of Proposition 8.3.

## 9. ABUNDANCE THEOREM FOR $\mathbb{R}$ -DIVISORS ( $k \neq \overline{\mathbb{F}}_p$ )

In this section, we establish the abundance theorem in the case where  $\Delta$  is an  $\mathbb{R}$ -boundary. We fix the following notations.



**Notation 9.1.** Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface and let  $\Delta$  be an effective  $\mathbb{R}$ -divisor. Let  $\Delta = \sum b_i B_i$  be its prime decomposition. Set  $B'_i := \lceil b_i \rceil B_i$  and

$$\mathcal{L} := \{B \in \sum \mathbb{R} B_i \mid 0 \leq B \leq \sum B'_i\}.$$

Let

$$M(X, \mathcal{L}) := \max(\{3\} \cup \{-(K_X + B'_\mu) \cdot B_\mu\})$$

where  $B_\mu$  ranges over the prime components of  $\Delta$  with  $B_\mu^2 < 0$ , respectively.

Lemma 9.2 and Proposition 9.4 play key roles in this section. The arguments are extracted from [Birkar, Section 3].

**Lemma 9.2.** *If  $R$  is an extremal ray of  $\overline{NE}(X)$  spanned by a curve, then there exists a curve  $C$  such that  $R = \mathbb{R}_{\geq 0}[C]$  and  $-(K_X + B) \cdot C \leq M(X, \mathcal{L})$  for all  $B \in \mathcal{L}$ .*

*Proof.* Let  $H$  be an ample line bundle on  $X$ . Take a curve  $C$  with

$$R = \mathbb{R}_{\geq 0}[C] \text{ and } H \cdot C = \min\{H \cdot D\}$$

where  $D$  ranges over curves generating  $R$ . We want to prove that  $C$  satisfies the desired condition. Set  $B \in \mathcal{L}$ . If  $-(K_X + B) \cdot C \leq 0$ , then there is nothing to prove. Thus we may assume that  $-(K_X + B) \cdot C > 0$ . This means that  $R$  is a  $(K_X + B)$ -negative extremal ray. Then, by Theorem 4.4 and Remark 4.5, there exists a curve  $C'$  such that  $R = \mathbb{R}_{\geq 0}[C']$  and

$$-(K_X + B) \cdot C' \leq L(X, B) = \max(\{3\} \cup \{-(K_X + B) \cdot B_\mu\})$$

where  $B_\mu$  ranges over the prime components of  $\Delta$  with  $B_\mu^2 < 0$ . Here, by the definition of  $M(X, \mathcal{L})$ , we have  $L(X, B) \leq M(X, \mathcal{L})$ . Thus we obtain

$$-(K_X + B) \cdot C' \leq M(X, \mathcal{L}).$$

By

$$\frac{-(K_X + B) \cdot C}{H \cdot C} = \frac{-(K_X + B) \cdot C'}{H \cdot C'},$$

we have

$$\begin{aligned} -(K_X + B) \cdot C &= (-(K_X + B) \cdot C') \frac{H \cdot C}{H \cdot C'} \\ &\leq -(K_X + B) \cdot C' \\ &\leq M(X, \mathcal{L}). \end{aligned}$$

This completes the proof.  $\square$

**Definition 9.3.** For an  $\mathbb{R}$ -divisor  $B$  and for its prime decomposition  $B = \sum r_i B_i$ , we define

$$\|B\| := \left( \sum |r_i|^2 \right)^{\frac{1}{2}}$$

where  $|r_i|$  is the absolute value of  $r_i$ .

**Proposition 9.4.** *Let  $\Gamma$  be a  $\mathbb{Q}$ -divisor on  $X$ . Let  $M$  be a positive real number.*

- (1) *There exists a positive real number  $\epsilon$  depending on  $X$ ,  $\Delta$ ,  $\Gamma$  and  $M$ , which satisfy the following property. Let  $C$  be a curve on  $X$  such that  $-(K_X + \Gamma + B) \cdot C \leq M$  for all  $B \in \mathcal{L}$ . If  $(K_X + \Gamma + \Delta) \cdot C > 0$ , then  $(K_X + \Gamma + \Delta) \cdot C > \epsilon$ .*
- (2) *There exists a positive real number  $\delta$ , depending on  $X$ ,  $\Delta$ ,  $\Gamma$  and  $M$ , which satisfy the following property. If a curve  $C'$  in  $X$  and an  $\mathbb{R}$ -divisor  $B_0 \in \mathcal{L}$  satisfy  $\|B_0 - \Delta\| < \delta$ ,  $(K_X + \Gamma + B_0) \cdot C' \leq 0$  and  $-(K_X + \Gamma + B) \cdot C' \leq M$  for all  $B \in \mathcal{L}$ , then  $(K_X + \Gamma + \Delta) \cdot C' \leq 0$ .*
- (3) *Let  $\{C_t\}_{t \in T}$  be a set of curves such that  $-(K_X + \Gamma + B) \cdot C_t \leq M$  for all  $B \in \mathcal{L}$ . Then, the set*

$$\mathcal{N}_T(\Gamma) := \{B \in \mathcal{L} \mid (K_X + \Gamma + B) \cdot C_t \geq 0 \text{ for any } t \in T\}$$

*is a rational polytope.*

*Proof.* Let  $V_1, \dots, V_r$  be the vertices of  $\mathcal{L}$ . Note that for every  $B \in \mathcal{L}$ , we have

$$B = \sum v_i V_i$$

for some real numbers  $v_i$  with  $0 \leq v_i \leq 1$  and  $\sum v_i = 1$ .

- (1) We can write  $\Delta := \sum v_i V_i$  as above. Then we have

$$(K_X + \Gamma + \Delta) \cdot C = \sum v_i (K_X + \Gamma + V_i) \cdot C.$$

Suppose  $(K_X + \Gamma + \Delta) \cdot C < 1$ . Then we have

$$\begin{aligned} v_i (K_X + \Gamma + V_i) \cdot C &< 1 - \sum_{j \neq i} v_j (K_X + \Gamma + V_j) \cdot C \\ &\leq 1 + \sum_{j \neq i} v_j M \\ &\leq 1 + (r-1)M. \end{aligned}$$

Thus, if  $v_i \neq 0$ , then we obtain

$$-M \leq (K_X + \Gamma + V_i) \cdot C < \frac{1}{v_i} (1 + (r-1)M).$$

Since  $X$  is  $\mathbb{Q}$ -factorial, the  $\mathbb{Q}$ -divisor  $K_X + \Gamma + V_i$  is  $\mathbb{Q}$ -Cartier. This means that there are only finitely many possibilities for the number  $(K_X + \Gamma + V_i) \cdot C$ .

Thus, if  $(K_X + \Gamma + \Delta) \cdot C < 1$ , then there are only finitely many possibilities for the number  $(K_X + \Gamma + \Delta) \cdot C = \sum v_i (K_X + \Gamma + V_i) \cdot C$ . Therefore we can find the desired number  $\epsilon$ .

(2) Suppose that the statement is not true. Then, for an arbitrary positive real number  $\delta$ , there exist a curve  $C'$  and an  $\mathbb{R}$ -divisor  $B_0 \in \mathcal{L}$  which satisfy  $\|B_0 - \Delta\| < \delta$ ,  $(K_X + \Gamma + B_0) \cdot C' \leq 0$ ,  $-(K_X + \Gamma + B) \cdot C' \leq M$  for all  $B \in \mathcal{L}$  and  $(K_X + \Gamma + \Delta) \cdot C' > 0$ . Set  $\delta := 1/m$  for any  $m \in \mathbb{Z}_{>0}$ . Then we obtain an infinite sequence of curves  $C_m$  and  $B_m \in \mathcal{L}$  which satisfy

$$(K_X + \Gamma + B_m) \cdot C_m \leq 0,$$

$$-(K_X + \Gamma + B) \cdot C_m \leq M \text{ for all } B \in \mathcal{L} \text{ and}$$

$$(K_X + \Gamma + \Delta) \cdot C_m > 0$$

and  $\|B_m - \Delta\|$  converges to zero. Let  $\Delta = \sum v_i V_i$  and  $B_m = \sum v_{i,m} V_i$  as above. Then we see  $v_i = \lim v_{i,m}$ . Here, for each  $j$ , the set  $\{(K_X + \Gamma + V_j) \cdot C_m\}_m$  has a lower bound  $-M$ .

Let us show that, if  $v_j \neq 0$ , then the set  $\{(K_X + \Gamma + V_j) \cdot C_m\}_m$  has an upper bound. Since  $0 < v_j = \lim v_{j,m}$ , we may assume  $v_{j,m} > 0$  for all  $m$  by replacing the sequence with a suitable sub-sequence. By the inequality

$$0 \geq (K_X + \Gamma + B_m) \cdot C_m = \sum v_{i,m} (K_X + \Gamma + V_i) \cdot C_m,$$

we have

$$\begin{aligned} (K_X + \Gamma + V_j) \cdot C_m &\leq \frac{1}{v_{j,m}} \left( - \sum_{i \neq j} v_{i,m} (K_X + \Gamma + V_i) \cdot C_m \right) \\ &\leq \frac{1}{v_{j,m}} \left( \sum_{i \neq j} v_{i,m} M \right) \\ &\leq \frac{1}{v_{j,m}} (r-1) M \end{aligned}$$

Since the set  $\{1/v_{j,m}\}_m$  has an upper bound, the set  $\{(K_X + \Gamma + V_j) \cdot C_m\}_m$  also has an upper bound. This is what we want to show.

Then we have

$$\begin{aligned}
& (K_X + \Gamma + B_m) \cdot C_m \\
= & (K_X + \Gamma + \Delta) \cdot C_m + \sum (v_{i,m} - v_i)(K_X + \Gamma + V_i) \cdot C_m \\
> & \epsilon + \sum (v_{i,m} - v_i)(K_X + \Gamma + V_i) \cdot C_m \\
= & \epsilon + \sum_{v_i \neq 0} (v_{i,m} - v_i)(K_X + \Gamma + V_i) \cdot C_m + \sum_{v_i = 0} v_{i,m}(K_X + \Gamma + V_i) \cdot C_m \\
\geq & \epsilon + \sum_{v_i \neq 0} (v_{i,m} - v_i)(K_X + \Gamma + V_i) \cdot C_m + \sum_{v_i = 0} v_{i,m}(-M) \\
> & 0.
\end{aligned}$$

The first inequality follows from (1). The third inequality follows when  $m \gg 0$ . Note that, if  $v_i \neq 0$ , then the set  $\{(K_X + \Gamma + V_i) \cdot C_m\}_i$  is bounded from the both sides. This is a contradiction.

(3) We may assume that for each  $t \in T$  there exists  $B \in \mathcal{L}$  with  $(K_X + \Gamma + B) \cdot C_t < 0$ .

We see that  $\mathcal{N}_T(\Gamma)$  is a compact set. Then, by (2) and by the compactness of  $\mathcal{N}_T(\Gamma)$ , there exist  $\Delta_1, \dots, \Delta_n \in \mathcal{N}_T(\Gamma)$  and  $\delta_1, \dots, \delta_n > 0$  such that  $\mathcal{N}_T(\Gamma)$  is covered by  $\mathcal{B}_i := \{B \in \mathcal{L} \mid \|B - \Delta_i\| < \delta_i\}$  and that if  $B \in \mathcal{B}_i$  with  $(K_X + \Gamma + B) \cdot C_t < 0$  for some  $t \in T$ , then  $(K_X + \Gamma + \Delta_i) \cdot C_t = 0$ . Set

$$T_i := \{t \in T \mid (K_X + \Gamma + B) \cdot C_t < 0 \text{ for some } B \in \mathcal{B}_i\}.$$

Then, for every  $t \in T_i$ , we have  $(K_X + \Gamma + \Delta_i) \cdot C_t = 0$ .

Here, we prove

$$\mathcal{N}_T(\Gamma) = \bigcap \mathcal{N}_{T_i}(\Gamma).$$

The inclusion  $\mathcal{N}_T(\Gamma) \subset \bigcap \mathcal{N}_{T_i}(\Gamma)$  is obvious. Thus we want to prove  $\mathcal{N}_T(\Gamma) \supset \bigcap \mathcal{N}_{T_i}(\Gamma)$ . Let  $B \notin \mathcal{N}_T(\Gamma)$ . Since  $\mathcal{N}_T(\Gamma)$  is compact, we can find an element  $B' \in \mathcal{N}_T(\Gamma)$  with

$$\|B' - B\| = \min\{\|B^* - B\| \mid B^* \in \mathcal{N}_T(\Gamma)\}.$$

Here we have  $B' \in \mathcal{B}_i$  for some  $i$ . Since  $\mathcal{B}_i \cap \overline{BB'}$  is an open subset of  $\overline{BB'}$  where  $\overline{BB'}$  is the line segment, we have an element  $B''$  such that  $B'' \in \mathcal{B}_i \cap \overline{BB'}$ ,  $B'' \neq B$  and  $B'' \neq B'$ . This means that there is a real number  $\beta$  with  $0 < \beta < 1$  such that

$$\beta B + (1 - \beta)B' = B''.$$

We obtain

$$\beta(K_X + \Gamma + B) + (1 - \beta)(K_X + \Gamma + B') = K_X + \Gamma + B''.$$

Moreover, we see that  $B'' \notin \mathcal{N}_T(\Gamma)$ . Here, since  $B'' \in \mathcal{B}_i \setminus \mathcal{N}_T(\Gamma)$ , we have  $(K_X + \Gamma + B'') \cdot C_t < 0$  for some  $t \in T_i$ . Thus we obtain the following inequality

$$\begin{aligned} & \beta(K_X + \Gamma + B) \cdot C_t \\ = & (K_X + \Gamma + B'') \cdot C_t - (1 - \beta)(K_X + \Gamma + B') \cdot C_t \\ < & -(1 - \beta)(K_X + \Gamma + B') \cdot C_t \\ \leq & 0. \end{aligned}$$

Therefore we have  $(K_X + \Gamma + B) \cdot C_t < 0$ . This means  $B \notin \mathcal{N}_{T_i}(\Gamma)$ .

Therefore it is enough to prove that each  $\mathcal{N}_{T_i}(\Gamma)$  is a rational polytope. By replacing  $T$  with  $T_i$ , we may assume that there exists  $\Delta_0 \in \mathcal{N}_T(\Gamma)$  such that  $(K_X + \Gamma + \Delta_0) \cdot C_t = 0$  for every  $t \in T$ . If  $\dim \mathcal{L} = 0$ , there is nothing to prove. Thus we assume  $\dim \mathcal{L} > 0$ . Let  $\mathcal{L}^1, \dots, \mathcal{L}^u$  be the proper faces of  $\mathcal{L}$  whose codimensions are one. Note that, for every  $1 \leq u' \leq u$ , there exists a positive integer  $i'$  such that

$$(I) \quad \mathcal{L}^{u'} = \{B \in \sum_{i \neq i'} \mathbb{R}B_i \mid 0 \leq B \leq \sum_{i \neq i'} B'_i\}$$

or that

$$(II) \quad \mathcal{L}^{u'} = B'_{i'} + \{B \in \sum_{i \neq i'} \mathbb{R}B_i \mid 0 \leq B \leq \sum_{i \neq i'} B'_i\}.$$

Let us prove that each  $\mathcal{N}_T^{u'}(\Gamma) := \mathcal{N}_T(\Gamma) \cap \mathcal{L}^{u'}$  is a rational polytope. If  $\mathcal{L}^{u'}$  satisfies the above equation (I), then we see that

$$\mathcal{N}_T^{u'}(\Gamma) = \{B \in \mathcal{L}^{u'} \mid (K_X + \Gamma + B) \cdot C_t \geq 0 \text{ for any } t \in T\}.$$

Hence  $\mathcal{N}_T^{u'}(\Gamma)$  is a rational polytope by the induction hypothesis. Thus assume that  $\mathcal{L}^{u'}$  satisfies the above equation (II). Set  $\mathcal{L}_0^{u'} := \{B \in \sum_{i \neq i'} \mathbb{R}B_i \mid 0 \leq B \leq \sum_{i \neq i'} B'_i\}$ . The equation (II) implies

$$\mathcal{L}^{u'} = B'_{i'} + \mathcal{L}_0^{u'}.$$

Then we see that

$$\begin{aligned} & \mathcal{N}_T^{u'}(\Gamma) \\ = & \{B \in \mathcal{L}^{u'} \mid (K_X + \Gamma + B) \cdot C_t \geq 0 \text{ for any } t \in T\} \\ = & B'_{i'} + \{B_0 \in \mathcal{L}_0^{u'} \mid (K_X + \Gamma + B'_{i'} + B_0) \cdot C_t \geq 0 \text{ for any } t \in T\}. \end{aligned}$$

For all  $B_0 \in \mathcal{L}_0^{u'}$ , we have the following inequality

$$-(K_X + \Gamma + B'_{i'} + B_0) \cdot C_t \leq M.$$

Thus the set

$$\begin{aligned} & \mathcal{N}_T(\mathcal{L}_0^{u'}, \Gamma + B_{i'}) \\ &:= \{B_0 \in \mathcal{L}_0^{u'} \mid (K_X + \Gamma + B_{i'} + B_0) \cdot C_t \geq 0 \text{ for any } t \in T\} \end{aligned}$$

is a rational polytope by the induction hypothesis. Therefore  $\mathcal{N}_T^{u'}(\Gamma)$  is also a rational polytope and this is what we want to show.

Here, take an arbitrary element  $B \in \mathcal{N}_T(\Gamma)$  with  $B \neq \Delta_0$ . Then we can find  $B' \in \mathcal{L}^{u'}$  for some  $1 \leq u' \leq u$  such that  $B$  is on the line segment defined by  $\Delta_0$  and  $B'$ . Since  $(K_X + \Gamma + \Delta_0) \cdot C_t = 0$  for all  $t \in T$ , we have  $B' \in \mathcal{N}_T^{u'}(\Gamma)$ . Thus we see that  $\mathcal{N}_T(\Gamma)$  is the convex hull of  $\Delta_0$  and all the  $\mathcal{N}_T^{u'}(\Gamma)$ . Hence  $\mathcal{N}_T(\Gamma)$  is a rational polytope.  $\square$

**Corollary 9.5.** *Let  $\{R_t\}_{t \in T}$  be a family of extremal rays of  $\overline{NE}(X)$  spanned by curves. Then the set*

$$\mathcal{N}_T := \{B \in \mathcal{L} \mid (K_X + B) \cdot R_t \geq 0 \text{ for any } t \in T\}$$

*is a rational polytope.*

*Proof.* By Lemma 9.2, for every  $t \in T$ , there exists a curve  $C_t$  such that  $R_t = \mathbb{R}_{\geq 0}[C_t]$  and  $-(K_X + B) \cdot C_t \leq M(X, \mathcal{L})$  for all  $B \in \mathcal{L}$ . Let  $\Gamma := 0$  and  $M := M(X, \mathcal{L})$ . Then, we can apply Proposition 9.4. Therefore, the set  $\mathcal{N}_T = \mathcal{N}_T(0)$  is a rational polytope.  $\square$

Now, we prove the abundance theorem with  $\mathbb{R}$ -coefficients.

**Theorem 9.6.** *Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface over  $k$  and let  $\Delta$  be an  $\mathbb{R}$ -boundary. If  $k$  is not the algebraic closure of a finite field and  $K_X + \Delta$  is nef, then  $K_X + \Delta$  is semi-ample.*

*Proof.* Let  $\{R_t\}_{t \in T}$  be the set of all the extremal rays of  $\overline{NE}(X)$  spanned by curves. Then

$$\mathcal{N}_T := \{B \in \mathcal{L} \mid (K_X + B) \cdot R_t \geq 0 \text{ for every } t \in T\}$$

is a rational polytope by Corollary 9.5. Moreover, by Theorem 4.4, we see that

$$\begin{aligned} \mathcal{N}_T &= \{B \in \mathcal{L} \mid (K_X + B) \cdot R_t \geq 0 \text{ for every } t \in T\} \\ &= \{B \in \mathcal{L} \mid K_X + B \text{ is nef}\}. \end{aligned}$$

Since  $\Delta \in \mathcal{N}_T$ , we can find  $\mathbb{Q}$ -divisors  $\Delta_1, \dots, \Delta_l$  such that  $\Delta_i \in \mathcal{N}_T$  for all  $i$  and that  $\sum r_i \Delta_i = \Delta$  where positive real numbers  $r_i$  satisfy  $\sum r_i = 1$ . Thus we have

$$K_X + \Delta = \sum r_i (K_X + \Delta_i)$$

and  $K_X + \Delta_i$  is nef. By Theorem 8.4,  $K_X + \Delta_i$  is semi-ample.  $\square$

**Part 3. Normal surfaces over  $\overline{\mathbb{F}}_p$** **10. CONTRACTION PROBLEM**

In this section, let  $k$  be an arbitrary algebraically closed field and  $\text{char } k = p \geq 0$ . As the introduction of this part, we consider the following question.

**Question 10.1** (Contraction problem). Let  $X$  be a smooth projective surface over  $k$  and let  $C$  be a curve in  $X$ . If  $C^2 < 0$ , then  $C$  is contractable? (i.e. Does there exist a birational morphism  $f : X \rightarrow Y$  to an algebraic surface  $Y$  such that  $f(C')$  is one point iff  $C' = C$  for every curve  $C'$ ?)

**Answer 10.2.** *If  $k \neq \overline{\mathbb{F}}_p$ , then the answer to Question 10.1 is NO in general.*

We only recall the method of its construction. For more details, see [Hartshorne, Example 5.7.3].

*Construction.* If we obtain an elliptic curve  $C_0$  in  $\mathbb{P}^2$  with  $\text{rank} \geq 10$ , then we can construct a counter-example as follows. There are 10 points in  $C_0$  which are linearly independent. Blow-up  $\mathbb{P}^2$  at these 10 points. The proper transform  $C$  of  $C_0$  is not contractable.  $\square$

By Fact 2.4, if  $k \neq \overline{\mathbb{F}}_p$ , then we can use this construction. On the other hand, if  $k = \overline{\mathbb{F}}_p$ , then we have the opposite answer.

**Answer 10.3** ([Artin]). *If  $k = \overline{\mathbb{F}}_p$ , then the answer to Question 10.1 is YES.*

To see this answer and its mechanism of this proof, we divide the verification into small pieces and prove more general following result.

**Proposition 10.4.** *Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface over  $k$  and let  $C$  be a curve in  $X$ .*

- (1) *If  $C^2 < 0$ , then there exists a nef and big divisor  $G$  such that  $G \cdot C' = 0$  iff  $C' = C$  for any curve  $C'$  in  $X$ .*
- (2) *If the restriction  $G|_C$  of the divisor  $G$  in (1) is a torsion and  $\text{char } k = p > 0$ , then  $G$  is semi-ample.*
- (3) *If  $k = \overline{\mathbb{F}}_p$ , then  $G|_C$  is torsion.*

*Proof.* (1) Let  $H$  be an ample divisor on  $X$ . We define a  $\mathbb{Q}$ -divisor  $G$  and  $q \in \mathbb{Q}_{>0}$  by  $G = H + qC$  and  $G \cdot C = 0$ . It is easy to check that  $G$  satisfies the above conditions.

(2) Since  $p > 0$ , we can use Keel's result (Theorem 2.2). Therefore, the semi-amplicity of  $G$  is equivalent to the semi-amplicity of  $G|_C$ . But

$G|_C$  is a torsion by the assumption. Thus,  $G$  is semi-ample.

(3) This is an immediate consequence of Corollary 2.5.  $\square$

## 11. $\mathbb{Q}$ -FACTORIALITY

In this section, we prove the following two theorems.

**Theorem 11.1.** *If  $X$  is a normal surface over  $\overline{\mathbb{F}}_p$ , then  $X$  is  $\mathbb{Q}$ -factorial.*

**Theorem 11.2.** *Let  $f : X \rightarrow Y$  be a proper birational morphism between normal surfaces over  $\overline{\mathbb{F}}_p$ , then  $f$  factors into contractions of one curve. More precisely, there exist proper birational morphisms such that each  $g_i : X_i \rightarrow X_{i+1}$  is a proper birational morphism between normal surfaces such that  $\text{Ex}(g_i)$  is an irreducible curve*

The following lemma is the key in this section.

**Lemma 11.3.** *Let  $f : X \rightarrow Y$  be a proper birational morphism over  $\overline{\mathbb{F}}_p$  from a normal  $\mathbb{Q}$ -factorial surface  $X$  to a normal surface  $Y$ . Let  $\text{Ex}(f) = C_1 \cup \cdots \cup C_r$ .*

- (1) *There exists a proper birational morphism  $g : X \rightarrow Z$  to a normal surface  $Z$  such that  $\text{Ex}(g) = C_1$  and  $g(C_1)$  is one point.*
- (2) *The morphism  $f$  factors through  $Z$ .*
- (3)  *$Z$  is  $\mathbb{Q}$ -factorial.*

*Proof.* (1) If  $X$  and  $Y$  are proper, then the assertion follows from Proposition 10.4. Note that proper  $\mathbb{Q}$ -factorial surfaces are projective (cf. [Fujino2, Lemma 2.2]). In general case, take the Nagata compactification. Note that normality and  $\mathbb{Q}$ -factoriality may break up by compactification. But by taking the normalization and the resolution of the locus of  $\overline{X} \setminus X$ , we may assume that these assumption.

(2) This is obvious.

(3) The assertion immediately follows from Proposition 6.6, Proposition 6.7 and Proposition 10.4.  $\square$

**Corollary 11.4.** *Let  $f : X \rightarrow Y$  be a proper birational morphism over  $\overline{\mathbb{F}}_p$  from a normal  $\mathbb{Q}$ -factorial surface  $X$  to a normal surface  $Y$ . Then  $Y$  is  $\mathbb{Q}$ -factorial.*

*Proof.* By using the above lemma repeatedly,  $f$  is factored into contractions of one curve and  $\mathbb{Q}$ -factoriality of  $X$  descends to  $Y$ .  $\square$

By the same argument, Theorem 11.2 follows from Theorem 11.1. Thus we only prove Theorem 11.1.



*Proof of Theorem 11.1.* Let  $f : X' \rightarrow X$  be its resolution of singularities. Of course  $X'$  is  $\mathbb{Q}$ -factorial. Therefore  $X$  is also  $\mathbb{Q}$ -factorial by Corollary 11.4.  $\square$

**Remark 11.5.** Theorem 11.1 follows from [Bădescu, Corollary 14.22] and [Matsumura, (24.E)].

## 12. THEOREMS IN PART 2

In this section, we establish the theorems, which we discussed in Part 2, over  $\overline{\mathbb{F}}_p$  under much weaker assumptions.

**Theorem 12.1** (Contraction theorem). *Let  $X$  be a projective normal surface over  $\overline{\mathbb{F}}_p$  and let  $\Delta$  be an effective  $\mathbb{R}$ -divisor. Let  $R = \mathbb{R}_{\geq 0}[C]$  be a  $(K_X + \Delta)$ -negative extremal ray. Then there exists a surjective morphism  $\phi_R : X \rightarrow Y$  to a projective variety  $Y$  with the following properties: (1)-(4).*

- (1) *Let  $C'$  be a curve on  $X$ . Then  $\phi_R(C')$  is one point iff  $[C'] \in R$ .*
- (2)  *$(\phi_R)_*(\mathcal{O}_X) = \mathcal{O}_Y$ .*
- (3) *If  $L$  is an invertible sheaf with  $L \cdot C = 0$ , then  $nL = (\phi_R)^*L_Y$  for some invertible sheaf  $L_Y$  on  $Y$  and for some positive integer  $n$ .*
- (4)  *$\rho(Y) = \rho(X) - 1$ .*

*Proof.* If  $C^2 \geq 0$ , then we have

$$K_X \cdot C \leq (K_X + \Delta) \cdot C < 0.$$

Then we can apply Theorem 6.2. Thus we may assume  $C^2 < 0$ . But this curve is contractable and the proofs of the remaining properties are the same as Theorem 6.2.  $\square$

**Theorem 12.2** (Finite generation theorem). *Let  $X$  be a projective normal surface over  $\overline{\mathbb{F}}_p$  and let  $D$  be a  $\mathbb{Q}$ -divisor. Then  $R(X, D) = \bigoplus_{m \geq 0} H^0(X, \lfloor mD \rfloor)$  is a finitely generated  $\overline{\mathbb{F}}_p$ -algebra.*

*Proof.* We may assume that  $\kappa(X, D) \geq 1$ . Then in particular  $D$  is effective. If there is a curve with  $D \cdot C < 0$ , then  $C^2 < 0$  and  $C$  is contractable. Let  $f : X \rightarrow Y$  be the contraction of  $C$ . Note that we obtain  $D = f^*f_*D + qC$ , for a positive rational number  $q$ . Therefore we may assume that  $D$  is nef. If  $\kappa(X, D) = 1$ , then  $D$  is semi-ample by Proposition 6.4. If  $\kappa(X, D) = 2$ , then  $D$  is semi-ample by the following proposition.  $\square$

**Proposition 12.3.** *Let  $X$  be a projective normal surface over  $\overline{\mathbb{F}}_p$ . If  $D$  is a nef and big  $\mathbb{Q}$ -divisor, then  $D$  is semi-ample.*

*Proof.* If there is a curve  $C$  such that  $D \cdot C = 0$ , then  $C^2 < 0$  and  $C$  is contractible. Let  $f : X \rightarrow Y$  be its contraction and  $f^*D_Y = D$ . It is sufficient to prove  $D_Y$  is semi-ample. Repeating the same procedure, we see that  $D$  is a pull-back of an ample divisor.  $\square$

**Theorem 12.4** (Non-vanishing theorem). *Let  $X$  be a projective normal surface over  $\overline{\mathbb{F}}_p$  and let  $\Delta$  be an effective  $\mathbb{Q}$ -divisor. If  $K_X + \Delta$  is nef, then  $\kappa(X, K_X + \Delta) \geq 0$*

The proof of this theorem heavily depends on the argument in [Mařek].

*Proof.* We may assume that  $X$  is smooth by replacing it with its minimal resolution.

**Step 1.** If  $\kappa(X, K_X) \geq 0$ , then  $\kappa(X, K_X + \Delta) \geq \kappa(X, K_X) \geq 0$ . Thus we may assume that  $\kappa(X, K_X) = -\infty$ .

**Step 2.** In this step, we show that we may assume  $K_X + \Delta$  is not numerically trivial and  $h^2(X, m(K_X + \Delta)) = 0$  for  $m \gg 0$ .

If  $K_X + \Delta$  is numerically trivial, then  $K_X + \Delta$  is a torsion by Fact 2.4. Thus we obtain  $n(K_X + \Delta) \sim 0$  for some integer  $n$  and  $\kappa(X, K_X + \Delta) = 0$ . Therefore we may assume that  $K_X + \Delta$  is not numerically trivial. Then we obtain  $h^2(X, m(K_X + \Delta)) = h^0(X, K_X - m(K_X + \Delta)) = 0$  for  $m \gg 0$ . (We have  $(K_X + \Delta) \cdot C > 0$  for some curve. Then there exists an ample divisor  $A$  and an effective divisor  $E$  such that  $A = C + E$ . By the nefness of  $K_X + \Delta$ , we obtain  $(K_X + \Delta) \cdot A > 0$ . Then since  $(K_X - m(K_X + \Delta)) \cdot A < 0$  for sufficiently large integer  $m$ , we obtain  $h^0(X, K_X - m(K_X + \Delta)) = 0$ .)

**Step 3.** In this step we show that we may assume  $(K_X + \Delta)^2 = 0$ .

Suppose the contrary, that is, suppose  $(K_X + \Delta)^2 > 0$ . Then  $K_X + \Delta$  is nef and big. Then we obtain  $h^0(X, m(K_X + \Delta)) > 0$  for some positive integer  $m$ , and  $\kappa(X, K_X + \Delta) \geq 0$ .

We consider the two cases:  $X$  is rational or irrational.

**Step 4.** In this step, we prove the assertion when  $X$  is rational.

Now  $\chi(\mathcal{O}_X) = 1$  because  $X$  is rational. Then, the Riemann–Roch theorem shows that

$$\begin{aligned} & h^0(X, m(K_X + \Delta)) \\ &= h^1(X, m(K_X + \Delta)) + 1 + \frac{1}{2}m(K_X + \Delta) \cdot (m(K_X + \Delta) - K_X) \end{aligned}$$

where  $m \gg 0$ . The right hand side is positive, because

$$\begin{aligned} & m(K_X + \Delta) \cdot (m(K_X + \Delta) - K_X) \\ &= m(K_X + \Delta) \cdot ((m-1)(K_X + \Delta) + \Delta) \geq 0 \end{aligned}$$

by the nefness of  $K_X + \Delta$ . This is what we want to show.

Thus we may assume that  $X$  is an irrational ruled surface. We divide the proof into three cases:  $(K_X + \Delta) \cdot K_X < 0$ ,  $(K_X + \Delta) \cdot K_X > 0$  and  $(K_X + \Delta) \cdot K_X = 0$ .

**Step 5.** We assume that  $X$  is irrational and  $(K_X + \Delta) \cdot K_X < 0$ .

By Step 2, Step 3 and the Riemann-Roch theorem,  $h^0(X, m(K_X + \Delta)) > 0$  for some large integer  $m$ . This is what we want to show.

**Step 6.** We assume that  $X$  is irrational and  $(K_X + \Delta) \cdot K_X > 0$ .

Since  $(K_X + \Delta)^2 = 0$  and  $(K_X + \Delta) \cdot K_X > 0$ , we obtain  $(K_X + \Delta) \cdot \Delta < 0$ . This contradicts the nefness of  $K_X + \Delta$ .

**Step 7.** We assume that  $X$  is irrational and  $(K_X + \Delta) \cdot K_X = 0$ .

We assume  $\kappa(X, K_X + \Delta) = -\infty$  and derive a contradiction. By  $(K_X + \Delta) \cdot K_X = 0$  and  $(K_X + \Delta)^2 = 0$ , we obtain  $(K_X + \Delta) \cdot \Delta = 0$ . Let  $C$  be an arbitrary prime component of  $\Delta$ . Since  $\Delta \neq 0$ , we can take such a curve. (Indeed, if  $\Delta = 0$ , then  $K_X$  is nef. This contradicts that  $X$  is a ruled surface.) By  $(K_X + \Delta) \cdot \Delta = 0$  and the nefness of  $K_X + \Delta$ , we have  $(K_X + \Delta) \cdot C = 0$ . By Fact 2.4, we obtain  $n_1(K_X + \Delta)|_C \sim 0$  for some  $n_1 \in \mathbb{Z}_{>0}$ . Then we get the following exact sequence:

$$0 \rightarrow \mathcal{O}_X(n_1 n_2(K_X + \Delta) - C) \rightarrow \mathcal{O}_X(n_1 n_2(K_X + \Delta)) \rightarrow \mathcal{O}_C \rightarrow 0$$

for every  $n_2 \in \mathbb{Z}_{>0}$ . Here we want to prove that, for every  $n_2 \gg 0$ ,

$$h^2(X, n_1 n_2(K_X + \Delta) - C) = 0.$$

By Serre duality, we obtain  $h^2(X, n_1 n_2(K_X + \Delta) - C) = h^0(X, K_X + C - n_1 n_2(K_X + \Delta))$ . This is zero, by the same argument as Step 2.

Fix  $n_2 \gg 0$  and let  $n := n_1 n_2$ . By  $h^2(X, n(K_X + \Delta) - C) = 0$ , we have a surjection  $H^1(X, n(K_X + \Delta)) \rightarrow H^1(C, \mathcal{O}_C)$ . This means

$$h^1(X, n(K_X + \Delta)) \geq h^1(C, \mathcal{O}_C).$$

On the other hand, by  $h^0(X, n(K_X + \Delta)) = h^2(X, n(K_X + \Delta)) = 0$  and the Riemann-Roch theorem,

$$\begin{aligned} -h^1(X, n(K_X + \Delta)) &= \chi(\mathcal{O}_X) + \frac{1}{2}n(K_X + \Delta) \cdot \{n(K_X + \Delta) - K_X\} \\ &= \chi(\mathcal{O}_X) = 1 - h^1(B, \mathcal{O}_B). \end{aligned}$$

where  $\pi : X \rightarrow B$  is the ruling. Hence we have

$$h^1(B, \mathcal{O}_B) - 1 = h^1(X, n(K_X + \Delta)) \geq h^1(C, \mathcal{O}_C).$$

This shows that  $C$  is in some fiber of  $\pi$ . In particular, for a smooth fiber  $F$ , we have  $C \cdot F = 0$ . Recall that  $C$  is an arbitrary prime component

of  $\Delta$ , then we obtain  $\Delta \cdot F = 0$ . Thus we have

$$0 \leq (K_X + \Delta) \cdot F = K_X \cdot F = -2.$$

This is a contradiction.  $\square$

**Theorem 12.5** (Abundance theorem). *Let  $X$  be a projective normal surface over  $\overline{\mathbb{F}}_p$  and let  $\Delta$  be an effective  $\mathbb{R}$ -divisor. If  $K_X + \Delta$  is nef, then  $K_X + \Delta$  is semi-ample.*

*Proof.* By the same proof as Theorem 9.6, we may assume that  $\Delta$  is a  $\mathbb{Q}$ -divisor. By Theorem 12.4, we have  $\kappa(X, K_X + \Delta) \geq 0$ . By Proposition 6.4 and Proposition 12.3, we may assume  $\kappa(X, K_X + \Delta) = 0$ . Then we can apply the argument of Step 1 and Step 2 in Theorem 8.5. By (1) of Proposition 8.3, we have  $\kappa(S, Y) = 1$  for indecomposable curves of canonical type  $Y$  in  $S$  over  $\overline{\mathbb{F}}_p$ . This contradicts  $Z \neq 0$  and  $\kappa(S, Z) = 0$ .  $\square$

As an immediate corollary, we obtain the following basepoint free theorem.

**Theorem 12.6** (Basepoint free theorem). *Let  $X$  be a projective normal surface over  $\overline{\mathbb{F}}_p$  and let  $D$  be a nef divisor. If  $\kappa(X, qD - K_X) \geq 0$  for some positive rational number  $q$ , then  $D$  is semi-ample.*

*Proof.* Take  $qD - K_X \sim_{\mathbb{Q}} \Delta$ . We obtain  $qD \sim_{\mathbb{Q}} K_X + \Delta$  and can apply the abundance theorem.  $\square$

### 13. EXAMPLES

In this section, let  $k$  be an algebraically closed field of arbitrary characteristic. We want to see the difference between  $k = \overline{\mathbb{F}}_p$  and  $k \neq \overline{\mathbb{F}}_p$  by looking at some examples.

**Example 13.1** (cf. Theorem 6.2 and Theorem 12.1). *If  $k \neq \overline{\mathbb{F}}_p$ , then there exist a smooth projective surface  $X$  over  $k$  and an elliptic curve  $C$  in  $X$  such that, for an arbitrary positive real number  $\epsilon$ ,  $(K_X + (1 + \epsilon)C) \cdot C < 0$ ,  $C^2 < 0$  and  $C$  is not contractable.*

*Construction.* Consider the Answer 10.2 and its construction. There exist a smooth projective surface  $X$  and an elliptic curve  $C$  in  $X$  such that  $C^2 = -1$  and  $C$  is not contractable. Moreover, we have

$$\begin{aligned} (K_X + (1 + \epsilon)C) \cdot C &= (K_X + C) \cdot C + \epsilon C \cdot C \\ &= \epsilon C \cdot C < 0. \end{aligned}$$

This is what we want to show.  $\square$

**Example 13.2** (cf. Theorem 8.1 and Theorem 12.4). *If  $k \neq \overline{\mathbb{F}}_p$ , then there exist a smooth projective surface  $X$  over  $k$  and curves  $C_1$  and  $C_2$  in  $X$  such that*

$$K_X + (1 + \epsilon)C_1 + (1 - \epsilon)C_2 \equiv 0 \quad \text{and} \\ \kappa(X, K_X + (1 + \epsilon)C_1 + (1 - \epsilon)C_2) = -\infty$$

for an arbitrary positive rational number  $\epsilon$ .

*Construction.* Let  $P := \mathbb{P}^1$  and let  $E$  be an arbitrary elliptic curve. Set  $X_0 := P \times E$ . We construct  $X$  by applying the elementary transform to  $\mathbb{P}^1$ -bundle  $X_0$  at appropriate two points. Let  $e_1$  and  $e_2$  be points in  $E$  which are linearly independent. Fix two different points  $p_1$  and  $p_2$  in  $P$  and set  $S_1 := \{p_1\} \times E$  and  $S_2 := \{p_2\} \times E$ . Then we see

$$K_{X_0} \sim_{\mathbb{Q}} -(1 + \epsilon)S_1 - (1 - \epsilon)S_2$$

for an arbitrary rational number  $\epsilon$ . Let  $x_1 := (p_1, e_1)$  and  $x_2 := (p_2, e_2)$ . We take the elementary transform of  $X_0$  at  $x_1$  and  $x_2$ , and obtain  $X$ . (First blowup at  $x_1$ . Then the proper transform of the fiber through  $x_1$  is a  $(-1)$ -curve. Second contract this  $(-1)$ -curve and we get another  $\mathbb{P}^1$ -bundle. Repeat the same thing to  $x_2$ .) Let  $C_1$  and  $C_2$  be the proper transforms of  $S_1$  and  $S_2$  respectively, and  $F_1$  and  $F_2$  be the fibers corresponding to  $x_1$  and  $x_2$  respectively. Then we see

$$K_X \sim_{\mathbb{Q}} -(1 + \epsilon)C_1 - (1 - \epsilon)C_2 - \epsilon F_1 + \epsilon F_2,$$

which implies

$$K_X + (1 + \epsilon)C_1 + (1 - \epsilon)C_2 \sim_{\mathbb{Q}} \epsilon(-F_1 + F_2).$$

This divisor is numerically trivial. Here we want to show that  $\kappa(X, -F_1 + F_2) = -\infty$ , that is,  $-F_1 + F_2$  is not a torsion. Consider the ruling  $\pi : X \rightarrow E$  and one of its sections  $\sigma : E \rightarrow X$ . Then we have  $F_1 = \pi^*e_1$  and  $F_2 = \pi^*e_2$ . Linear independence of  $e_1$  and  $e_2$  shows

$$\mathcal{O}_X(n(-F_1 + F_2)) \not\simeq \mathcal{O}_X.$$

Indeed, if  $\mathcal{O}_X(n(-F_1 + F_2)) \simeq \mathcal{O}_X$ , then we have  $\pi^*\mathcal{O}_E(n(-e_1 + e_2)) \simeq \mathcal{O}_X$ . Then, we obtain

$$\mathcal{O}_E(n(-e_1 + e_2)) \simeq \sigma^*\pi^*\mathcal{O}_E(n(-e_1 + e_2)) \simeq \sigma^*\mathcal{O}_X \simeq \mathcal{O}_E.$$

This is a contradiction. □

**Example 13.3** (cf. Theorem 8.4 and Theorem 12.5). *If  $k \neq \overline{\mathbb{F}}_p$ , then there exist a projective smooth surface  $X$  over  $k$  and an elliptic curve  $C$  in  $X$  such that, for an arbitrary positive rational number  $\epsilon$ ,  $K_X + (1 + \epsilon)C$  is nef,  $\kappa(X, K_X + (1 + \epsilon)C) \geq 0$  and  $K_X + (1 + \epsilon)C$  is not semi-ample.*

*Construction.* Set  $X_0 := \mathbb{P}^2$ . Let  $C_0$  be an arbitrary elliptic curve in  $X_0$  and let  $P_1, \dots, P_9$  be points in  $C_0$  which are linearly independent. Blowup at these nine points, then we obtain the surface  $X$  and let  $C$  be the proper transform of  $C_0$ . By  $K_{X_0} = -C_0$ , we have  $K_X = -C$ . Then

$$K_X + (1 + \epsilon)C = \epsilon C$$

is nef by  $C^2 = 0$ . It is obvious that  $\kappa(X, K_X + (1 + \epsilon)C) \geq 0$ . We prove that  $K_X + (1 + \epsilon)C$  is not semi-ample. It is sufficient to prove  $\kappa(X, C) = 0$ . Suppose the contrary, that is, suppose  $\kappa(X, C) \geq 1$ . Then we obtain  $nC \sim D$  for some non-zero effective divisor  $D$  with  $C \not\subset \text{Supp} D$ . Since  $C \cdot D = 0$ ,  $\text{Supp}(f_*(D)|_{C_0})$  must be contained in  $P_1, \dots, P_9$ . This means  $n_1P_1 + \dots + n_9P_9 := f_*(D)|_{C_0} \sim 3nL|_{C_0}$ . Here  $L$  is a line in  $X_0$ . But this means  $n_1P_1 + \dots + n_9P_9 = 0$  in the group structure of  $C_0$ . This is a contradiction.  $\square$

## Part 4. Log canonical surfaces

### 14. LOG CANONICAL SINGULARITIES

In this section, we describe the log canonical singularities in surfaces by using the contraction theorem (Theorem 6.2).

**Definition 14.1.** We say a pair  $(X, \Delta)$  is a *log canonical surface* if a normal surface  $X$  and an  $\mathbb{R}$  divisor  $\Delta$  satisfy the following properties.

- (1)  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier.
- (2) For an arbitrary proper birational morphism  $f : Y \rightarrow X$  and the divisor  $\Delta_Y$  defined by

$$K_Y + \Delta_Y = f^*(K_X + \Delta),$$

the inequality  $\Delta_Y \leq 1$  holds.

- (3)  $\Delta$  is effective.

First, we pay attention to only one singular point.

**Definition 14.2.** We say  $(X, \Delta)$  is a *local situation of a log canonical surface* if it satisfies the following properties.

- (1) The pair  $(X, \Delta)$  is a log canonical surface.
- (2) There exists only one singular point  $x \in X$ .
- (3) All prime components of  $\Delta$  contain  $x$ .

**Theorem 14.3.** Let  $(X, \Delta)$  be a local situation of a log canonical surface and let  $f : Y \rightarrow X$  be the minimal resolution of  $X$ . Then, there exists a sequence of proper birational morphisms

$$f : Y =: Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} \dots \xrightarrow{f_{m-1}} Y_m =: Z \xrightarrow{g} X$$

with the following properties.

- (1) Each  $Y_i$  is a normal  $\mathbb{Q}$ -factorial surface.
- (2) Each  $f_i$  is a proper birational morphism and  $E_i := \text{Ex}(f_i)$  is an irreducible curve.
- (3) Each  $E_i$  satisfies  $(K_{Y_i} + E_i) \cdot E_i < 0$ .
- (4) One of (a) and (b) holds.
  - (a)  $g$  is an isomorphism.
  - (b)  $\Delta = 0$  and  $E := \text{Ex}(g)$  is an irreducible curve such that  $(K_Y + E) \cdot E = 0$ .

*Proof.* We assume that we obtain

$$f : Y =: Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{j-1}} Y_j \xrightarrow{G} X$$

such that each  $Y_i$  and each  $f_i$  satisfy (1), (2) and (3).

We prove that, if we can find a  $G$ -exceptional proper curve  $E_j$  such that  $(K_{Y_j} + E_j) \cdot E_j < 0$ , then we obtain a contraction of  $E_j$

$$f_j : Y_j \rightarrow Y_{j+1}$$

to a  $\mathbb{Q}$ -factorial surface  $Y_{j+1}$ . If  $X$  and  $Y_j$  are proper, then we obtain the required morphism  $f_j$  by Theorem 5.3, Proposition 6.5, Proposition 6.6 and Proposition 6.7. Note that proper  $\mathbb{Q}$ -factorial surface is projective (cf. [Fujino2, Lemma 2.2]). For the general case, take compactifications as follows. Let  $\overline{X}$  be a proper normal surface and let  $\overline{\Delta}$  be an  $\mathbb{R}$ -divisor on  $\overline{X}$  such that  $X \hookrightarrow \overline{X}$  is an open immersion,  $(\overline{X}, \overline{\Delta})$  is a local situation of log canonical surface and  $\overline{\Delta}|_X = \Delta$ . We define  $\overline{Y_j}$  by patching  $Y_j$  and  $\overline{X}$  along  $Y_j \setminus \text{Ex}(G) \simeq X \setminus \{x\}$ . Then,  $\overline{Y_j}$  is  $\mathbb{Q}$ -factorial. Thus, we can reduce the problem to the case where  $X$  and  $Y_j$  are proper.

If  $G$  is an isomorphism, then we obtain (a). Thus, we may assume  $G$  is not an isomorphism. Then, we can take a  $G$ -exceptional curve  $E_j$ . We obtain

$$(K_{Y_j} + E_j) \cdot E_j \leq (K_{Y_j} + \Delta_j) \cdot E_j = G^*(K_X + \Delta) \cdot E_j = 0$$

where  $\Delta_j$  is defined by  $K_{Y_j} + \Delta_j = G^*(K_X + \Delta)$ . We may assume  $(K_{Y_j} + E_j) \cdot E_j = 0$ . In this case, the coefficient of  $E_j$  in  $\Delta_j$  is one.

First, assume  $\text{Ex}(G)$  is reducible. Then, there exists a  $G$ -exceptional curve  $E'_j$  such that  $E_j \cap E'_j \neq \emptyset$ . Then, we have

$$(K_{Y_j} + E'_j) \cdot E'_j < (K_{Y_j} + \Delta_j) \cdot E'_j = G^*(K_X + \Delta_j) \cdot E'_j = 0.$$

This is what we want to show.

Second, assume  $E := \text{Ex}(G)$  is irreducible. Since  $(K_{Y_j} + E) \cdot E \leq 0$ , we consider the two cases:  $(K_{Y_j} + E) \cdot E < 0$  or  $(K_{Y_j} + E) \cdot E = 0$ . If

$(K_{Y_j} + E) \cdot E < 0$ , then this means (a). Assume  $(K_{Y_j} + E) \cdot E = 0$ . We show  $\Delta = 0$ . If  $\Delta \neq 0$ , then we have

$$(K_{Y_j} + E) \cdot E < (K_{Y_j} + \Delta_j) \cdot E = 0.$$

This means (b).  $\square$

This theorem teaches us that non- $\mathbb{Q}$ -factorial log canonical singularities are made by the case (b). Applying the same argument as above, we obtain the global version as follows.

**Theorem 14.4.** *Let  $(X, \Delta)$  be a log canonical surface and let  $f : Y \rightarrow X$  be the minimal resolution of  $X$ . Then, there exists a sequence of proper birational morphisms*

$$f : Y =: Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{m-1}} Y_m =: Z \xrightarrow{g} X$$

*with the following properties.*

- (1) *Each  $Y_i$  is a normal  $\mathbb{Q}$ -factorial surface.*
- (2) *Each  $f_i$  is a proper birational morphism and  $E_i := \text{Ex}(f_i)$  is an irreducible curve.*
- (3) *Each  $E_i$  satisfies  $(K_{Y_i} + E_i) \cdot E_i < 0$ .*
- (4) *One of (a) and (b) holds.*
  - (a)  *$g$  is an isomorphism.*
  - (b)  *$g(\text{Ex}(g)) \cap \text{Supp} \Delta = \emptyset$  and, for every point  $Q \in g(\text{Ex}(g))$ ,  $g^{-1}(Q) =: E$  is a proper irreducible curve such that  $(K_Y + E) \cdot E = 0$ .*

*In particular,  $K_X$  and all prime components of  $\Delta$  are  $\mathbb{Q}$ -Cartier.*

*Proof.* This follows from the same argument as Theorem 14.3.  $\square$

**Remark 14.5.** By Theorem 20.4, we see that  $Z$  have at worst rational singularities. But we do not use this fact in this paper.

## 15. MINIMAL MODEL THEORY FOR LOG CANONICAL SURFACES

In this section, we consider the minimal model theory for log canonical surfaces. We have already proved the cone theorem in Section 4. Thus let us consider the contraction theorem.

**Theorem 15.1** (Contraction theorem). *Let  $(X, \Delta)$  be a projective log canonical surface and let  $R = \mathbb{R}_{\geq 0}[C]$  be a  $(K_X + \Delta)$ -negative extremal ray. Then there exists a morphism  $\phi_R : X \rightarrow Y$  to a projective variety  $Y$  with the following properties: (1)-(5).*

- (1) *Let  $C'$  be a curve on  $X$ . Then  $\phi_R(C')$  is one point iff  $[C'] \in R$ .*
- (2)  *$\phi_*(\mathcal{O}_X) = \mathcal{O}_Y$*



- (3) If  $L$  is a line bundle with  $L \cdot C = 0$ , then  $nL = (\phi_R)^*L_Y$  for some line bundle  $L_Y$  on  $Y$  and for some positive integer  $n$ .
- (4)  $\rho(Y) = \rho(X) - 1$ .
- (5)  $(Y, (\phi_R)_*(\Delta))$  is a log canonical surface if  $\dim Y = 2$ .

*Proof of the case where  $C^2 > 0$ .* First, we prove that there exists a curve  $D$  in  $X$  such that  $D$  is Cartier,  $D$  is ample and  $\mathbb{R}_{\geq 0}[C] = \mathbb{R}_{\geq 0}[D]$ . Since  $X$  is a projective normal surface, we can apply Bertini's theorem. Then the complete linear system of a very ample divisor has a smooth member  $D$  such that  $D \cap \text{Sing}(X) = \emptyset$ . Note that  $D$  is a Cartier divisor. Let  $f : X' \rightarrow X$  be the minimal resolution and let  $D'$  be the proper transform of  $D$ . Since  $f^*(C)$  is a nef and big divisor, we obtain

$$nf^*(C) \sim D' + E$$

for some effective divisor  $E$  and some positive integer  $n$ . By sending this equation by  $f_*$ , we obtain

$$nC \sim D + f_*(E).$$

Since  $\mathbb{R}_{\geq 0}[C]$  is extremal, we have  $\mathbb{R}_{\geq 0}[C] = \mathbb{R}_{\geq 0}[D]$ . Thus we obtain  $\rho(X) = 1$ , because we can apply the same argument as the one in the proof of Theorem 6.2. Set  $Y := \text{Spec } k$ . Then  $\phi_R : X \rightarrow Y$  satisfies (1), (2) and (4). We want to prove (3). This follows from Lemma 15.2 because  $K_X + \Delta$  is anti-ample.  $\square$

**Lemma 15.2.** *Let  $(X, \Delta)$  be a projective log canonical surface. Let  $L$  be a nef line bundle such that  $L - (K_X + \Delta)$  is ample. Then,  $L$  is semi-ample.*

*Proof.* By Bertini's theorem, there exists a smooth curve  $C$  such that

$$nL - n(K_X + \Delta) \sim C,$$

$C \cap \text{Sing}(X) = \emptyset$  and  $C$  is not a component of  $\Delta$ . Let  $f : X' \rightarrow X$  be the minimal resolution and let  $C'$  be the proper transform of  $C$ . Then we obtain

$$nf^*(L) - nf^*(K_X + \Delta) \sim f^*(C).$$

Since  $f^*(C) = C'$ , we have

$$f^*(L) \sim_{\mathbb{Q}} K_{X'} + \Delta' + \frac{1}{n}C'$$

where  $\Delta'$  is defined by  $K_{X'} + \Delta' = f^*(K_X + \Delta)$ . Since  $\Delta' + (1/n)C'$  is a boundary,  $f^*(L)$  is semi-ample by Theorem 9.6 and Theorem 12.5. Therefore so is  $L$ .  $\square$

In the proof of the case where  $C^2 \leq 0$  in Theorem 6.2, we only use the assumption of  $\mathbb{Q}$ -factoriality in the form that  $K_X$  and  $C$  are  $\mathbb{Q}$ -Cartier and  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier. Since  $K_X$  is  $\mathbb{Q}$ -Cartier and  $K_X + \Delta$  are  $\mathbb{R}$ -Cartier by Theorem 14.4, it is sufficient to prove that  $C$  is  $\mathbb{Q}$ -Cartier.

*Proof of the case where  $C^2 = 0$ .* It is sufficient to prove that  $\mathbb{R}_{\geq 0}[C] = \mathbb{R}_{\geq 0}[D]$  for some  $\mathbb{Q}$ -Cartier curve  $D$ . Let  $f : X' \rightarrow X$  be the minimal resolution. Since  $f^*(C)^2 = 0$  and  $f^*(C) \cdot K_{X'} < 0$ , we obtain  $\kappa(X', f^*(C)) = 1$ . Therefore,  $f^*(C)$  is semi-ample by Proposition 6.4. We consider the fibration  $\pi : X' \rightarrow B$  obtained by the complete linear system  $|nf^*(C)|$  for some  $n \gg 0$ . For an arbitrary  $f$ -exceptional curve  $E$ , we have  $E \cdot f^*(C) = 0$ . This means that an arbitrary exceptional curve is in some fiber of  $\pi$ . Thus there exists an integral fiber  $D'$  of  $\pi$  with  $D' \cap \text{Ex}(f) = \emptyset$  by Proposition 15.3. This means that  $f(D') = D$  is Cartier and  $nC \equiv D$ . This is what we want to show.  $\square$

**Proposition 15.3.** *Let  $\pi : X \rightarrow S$  be a dominant morphism from a normal surface  $X$  to a curve  $S$  with  $\pi_*\mathcal{O}_X = \mathcal{O}_S$ . Then there exists a non-empty open subset  $S'$  in  $S$  such that all scheme-theoretic fibers of  $\pi|_{\pi^{-1}(S')} : \pi^{-1}(S') \rightarrow S'$  are integral.*

*Proof.* See, for example, [Bădescu, Corollary 7.3].  $\square$

For the proof of the case where  $C^2 < 0$ , we consider the relation between the non- $\mathbb{Q}$ -factorial log canonical singularities and extremal curves  $C$  with  $C^2 < 0$ . Since we want to prove that  $C$  is  $\mathbb{Q}$ -Cartier, it is necessary to consider the case where  $C$  passes through the singular points of (b) in Theorem 14.3. The following lemma teaches us these singularities are actually  $\mathbb{Q}$ -factorial.

**Lemma 15.4.** *Let  $(X, \Delta = 0)$  be a local situation of a log canonical surface and let  $x$  be the singular point of  $X$ . Assume that this singularity is (b) in Theorem 14.3. If a proper curve  $C$  in  $X$  satisfies  $C \cdot K_X < 0$ ,  $C^2 < 0$  and  $x \in C$ , then  $X$  is  $\mathbb{Q}$ -factorial.*

*Proof.* We use the notation in (b) of Theorem 14.3. It is sufficient to prove that  $E \simeq \mathbb{P}^1$  by Proposition 6.6 and Proposition 6.7. Let  $C_Z$  be the proper transform of  $C$ . Then, we obtain

$$\begin{aligned} C_Z^2 &\leq C_Z \cdot g^*(C) = C^2 < 0 \\ C_Z \cdot K_Z &\leq C_Z \cdot (K_Z + E) = C_Z \cdot g^*(K_X) = C \cdot K_X < 0. \end{aligned}$$

Thus we obtain  $C_Z \simeq \mathbb{P}^1$  and  $C_Z$  is a curve generating a  $K_Z$ -negative extremal ray. Let  $\phi : Z \rightarrow Z'$  be the contraction of  $C_Z$ . Since  $\phi :$

$E \rightarrow \phi(E) =: E'$  is a birational morphism, it is sufficient to prove that  $E' \simeq \mathbb{P}^1$ . We would like to prove

$$(K_{Z'} + E') \cdot E' < 0.$$

Let us consider the discrepancy  $d$  defined by

$$K_Z + E = \phi^*(K_{Z'} + E') + dC_Z.$$

Here, by taking the intersection with  $E$ , we obtain

$$0 = (K_{Z'} + E') \cdot E' + dC_Z \cdot E$$

by  $(K_Z + E) \cdot E = 0$ . By  $x \in C$ , we see that  $C_Z \cdot E$  is a positive number. Thus it is sufficient to prove that  $d$  is a positive number. The following inequality

$$0 > K_X \cdot C = g^*(K_X) \cdot C_Z = (K_Z + E) \cdot C_Z = dC_Z^2$$

shows that  $d$  is positive.  $\square$

**Proposition 15.5.** *Let  $(X, \Delta)$  be a log canonical surface. If a proper curve  $C$  in  $X$  satisfies  $C \cdot (K_X + \Delta) < 0$  and  $C^2 < 0$ , then  $C$  is  $\mathbb{Q}$ -Cartier.*

*Proof.* By Theorem 14.3 and Lemma 15.4,  $C$  passes through only  $\mathbb{Q}$ -factorial points.  $\square$

Thus we complete the proof of the Theorem 15.1. Next, we consider the abundance theorem. But this immediately follows from the  $\mathbb{Q}$ -factorial case.

**Theorem 15.6.** *Let  $(X, \Delta)$  be a proper log canonical surface. If  $K_X + \Delta$  is nef, then  $K_X + \Delta$  is semi-ample.*

*Proof.* Take the minimal resolution and apply Theorem 9.6 and Theorem 12.5.  $\square$

## Part 5. Relativization

### 16. RELATIVE CONE THEOREM

In this section, we consider the relativization of the cone theorem. But this is not difficult by the following proposition.

**Proposition 16.1.** *Let  $\pi : X \rightarrow S$  be a proper morphism from a normal surface  $X$  to a variety  $S$ . If  $\dim \pi(X) \geq 1$  where  $\pi(X)$  is the scheme-theoretic image of  $\pi$ , then we have*

$$\overline{NE}(X/S) = NE(X/S) = \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

Moreover, the Stein factorization  $\pi : X \xrightarrow{\theta} T \rightarrow S$  satisfies one of the following assertions:

- (1-irr) If  $\dim \pi(X) = 1$  and all fibers of  $\theta$  are irreducible, then  $NE(X/S) = \mathbb{R}_{\geq 0}[C]$  and  $C^2 = 0$ . In particular,  $\rho(X/S) = 1$ .
- (1-red) If  $\dim \pi(X) = 1$  and  $\theta$  has at least one reducible fiber, then each  $C_i$  has negative self-intersection number.
- (2) If  $\dim \pi(X) = 2$ , then each  $C_i$  has negative self-intersection number.

*Proof.* Note that  $\dim \pi(X) = \dim T$  and  $NE(X/S) = NE(X/T)$ .

(1-irr) All fibers are numerically equivalent. This is what we want to show.

(1-red) By Proposition 15.3, general fibers of  $\theta$  are irreducible. Therefore, there are only finitely many reducible fibers. Since all fibers are numerically equivalent,  $NE(X/S)$  is generated by the curves in the reducible fibers. Because all fibers of  $\theta$  is connected, curves in reducible fibers have negative self-intersection number.

(2) By  $\theta_*\mathcal{O}_X = \mathcal{O}_T$ , we see that  $\theta$  is birational. Since the exceptional locus is a closed set, there are only finitely many curves contracted by  $\theta$ . Each contracted curve has negative self-intersection number.  $\square$

Using this proposition, we obtain the following relative cone theorem.

**Theorem 16.2.** *Let  $\pi : X \rightarrow S$  be a projective morphism from a normal surface  $X$  to a variety  $S$ . Let  $\Delta$  be an effective  $\mathbb{R}$ -divisor such that  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier. Let  $\Delta = \sum b_i B_i$  be the prime decomposition. Let  $H$  be an  $\mathbb{R}$ -Cartier  $\pi$ -ample  $\mathbb{R}$ -divisor on  $X$ . Then the following assertions hold:*

- (1)  $\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X + \Delta \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$ .
- (2)  $\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X + \Delta + H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i]$ .
- (3) Each  $C_i$  in (1) and (2) is rational or  $C_i = B_j$  for some  $B_j$  with  $B_j^2 < 0$ .
- (4) There exists a positive integer  $L(X, S, \Delta)$  such that each  $C_i$  in (1) and (2) satisfies  $0 < -C_i \cdot (K_X + \Delta) \leq L(X, S, \Delta)$ .

*Proof.* If  $\dim \pi(X) = 0$ , then the assertion follows from Theorem 4.4. If  $\dim \pi(X) \geq 1$ , then the assertions (1), (2) and (4) immediately follow from Proposition 16.1. We prove (3). Let  $C$  be a  $(K_X + \Delta)$ -negative proper curve which generates an extremal ray and  $\pi(C)$  is one point. We may assume  $C \neq B_j$  for all  $B_j$  with  $B_j^2 < 0$ . Take the Stein factorization of  $\pi$ :

$$\pi : X \xrightarrow{\theta} T \rightarrow S.$$

Let us take the Nagata compactification of  $T$  and its normalization  $\overline{T}$ . Moreover, take the normalization  $\overline{X}$  of a compactification of  $X \rightarrow \overline{T}$ . We obtain the following commutative diagram.

$$\begin{array}{ccc} X & \xrightarrow[\text{immersion}]{\text{open}} & \overline{X} \\ \theta \downarrow & & \downarrow \overline{\theta} \\ T & \xrightarrow[\text{immersion}]{\text{open}} & \overline{T} \end{array}$$

In  $\overline{X}$ , we can apply (BBII) in the sense of Definition 3.6 to  $C$ . Then we obtain

$$p^n C \equiv_{\text{Mum}} \alpha C' + Z$$

for a positive integer  $n$ , a non-negative integer  $\alpha$ , a curve  $C'$  and a sum of rational curves  $Z$ . We consider the two cases:  $\dim T = 1$  and  $\dim T = 2$ .

Assume  $\dim T = 1$ . Take an ample divisor  $A$  on  $\overline{T}$ . Since  $C \cdot \overline{\theta}^* A = 0$ , the prime components of  $Z$  must be  $\overline{\theta}$ -vertical. In advance, let  $c_0 \in C$  be a point, in the notation of Definition 3.6, such that  $c_0$  is not contained in any curve  $C'' \neq C$  which is contained in the fiber containing  $C$ . Then, there exists a prime component  $Z_j$  of  $Z$  with  $c_0 \in Z_j$ . Here  $Z_j$  must be  $C$ . In particular,  $C$  is rational and this is what we want to show.

Assume  $\dim T = 2$ . Then,  $\overline{T}$  is a proper normal surface. Since  $\overline{\theta}_*(\alpha C' + Z) \equiv_{\text{Mum}} 0$ , each prime component of  $Z$  is  $\overline{\theta}$ -exceptional. The remaining proof is the same as the case of  $\dim T = 1$ .  $\square$

We give an upper bound  $L(X, S, \Delta)$  in the case where  $\Delta$  is an  $\mathbb{R}$ -boundary.

**Proposition 16.3.** *Let  $\pi : X \rightarrow S$  be a projective morphism from a normal surface  $X$  to a variety  $S$ . Let  $\Delta$  be an  $\mathbb{R}$ -boundary such that  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier. If  $R$  is a  $(K_X + \Delta)$ -negative extremal ray of  $\overline{NE}(X/S)$ , then  $R = \mathbb{R}_{\geq 0}[C]$  where  $C$  is a rational curve such that  $-(K_X + \Delta) \cdot C \leq 3$ .*

*Proof.* If  $\dim \pi(X) = 0$ , then the assertion follows from Proposition 4.6. Thus, we assume  $\dim \pi(X) \geq 1$ . We can write  $R = \mathbb{R}_{\geq 0}[C]$  for some curve  $C$ . We show  $C$  satisfies the desired properties. By  $\dim \pi(X) \geq 1$  and Proposition 16.1, we see  $C^2 \leq 0$ . Then, by Lemma 3.9, we have

$$-(K_X + \Delta) \cdot C \leq 2.$$

Since

$$(K_X + C) \cdot C \leq (K_X + \Delta) \cdot C < 0,$$

by Lemma 4.7, we see that  $C$  is rational.  $\square$

## 17. RELATIVE CONTRACTION THEOREM

In this section, we consider the relativization of the contraction theorem.

**Theorem 17.1.** *Let  $\pi : X \rightarrow S$  be a projective morphism from a normal surface  $X$  to a variety  $S$ . Let  $\Delta$  be an  $\mathbb{R}$  divisor. Moreover one of the following conditions holds:*

- (QF)  *$X$  is  $\mathbb{Q}$ -factorial and  $\Delta$  is an  $\mathbb{R}$ -boundary.*
- (FP)  *$k = \overline{\mathbb{F}}_p$  and  $\Delta$  is an effective  $\mathbb{R}$ -divisor.*
- (LC)  *$(X, \Delta)$  is a log canonical surface.*

Let  $R = \mathbb{R}_{\geq 0}[C]$  be a  $(K_X + \Delta)$ -negative extremal ray in  $\overline{NE}(X/S)$ . Then there exists a surjective  $S$ -morphism  $\phi_R : X \rightarrow Y$  to a variety  $Y$  projective over  $S$  with the following properties: (1)-(5).

- (1) *Let  $C'$  be a curve on  $X$ . Then  $\phi_R(C')$  is one point iff  $[C'] \in R$ .*
- (2)  *$(\phi_R)_*(\mathcal{O}_X) = \mathcal{O}_Y$ .*
- (3) *If  $L$  is an invertible sheaf with  $L \cdot C = 0$ , then  $nL = (\phi_R)^*L_Y$  for some invertible sheaf  $L_Y$  on  $Y$  and for some positive integer  $n$ .*
- (4)  *$\rho(Y/S) = \rho(X/S) - 1$ .*
- (5) *If  $\dim Y = 2$  then  $Y$  is  $\mathbb{Q}$ -factorial (resp.  $(Y, (\phi_R)_*(\Delta))$  is log canonical) in the case of (QF) (resp. (LC)).*

These three proofs of (QF), (FP) and (LC) are the same essentially. Thus we only prove the case when (QF).

*Proof.* Let  $\theta : X \rightarrow T$  be the Stein factorization of  $\pi$ . We see that  $\dim T = 0$ ,  $\dim T = 1$  or  $\dim T = 2$ . But the case  $\dim T = 0$  follows from Theorem 6.2. Thus we may assume that  $\dim T = 1$  or  $\dim T = 2$ .

Now let us take the compactification. First, take the Nagata compactification of  $T$  and its normalization  $\overline{T}$ . Second, take the compactification  $\overline{X}$  of  $X \rightarrow \overline{T}$ . Moreover, if necessary, replace it by its normalization and a resolution of the singular locus in  $\overline{X} \setminus X$ . We obtain the following commutative diagram.

$$\begin{array}{ccc} X & \xrightarrow[\text{immersion}]{\text{open}} & \overline{X} \\ \theta \downarrow & & \downarrow \overline{\theta} \\ T & \xrightarrow[\text{immersion}]{\text{open}} & \overline{T} \end{array}$$

Then,  $\overline{X}$  is projective normal  $\mathbb{Q}$ -factorial and  $\overline{T}$  is proper normal. Let  $\overline{\Delta}$  be the  $\mathbb{R}$ -boundary such that its restriction to  $X$  is  $\Delta$  and  $\overline{\Delta}$  has no prime components contained in  $\overline{X} \setminus X$ .

Assume  $C^2 < 0$ . This follows from Theorem 6.2 because  $C$  is a  $(K_{\overline{X}} + \overline{\Delta})$ -negative extremal curve in the cone of the absolute case  $\overline{NE}(\overline{X})$ .

Assume  $C^2 \geq 0$ . Then, by Proposition 16.1, we see  $\rho(X/T) = 1$  and  $\dim T = 1$ . Set  $Y := T$ . The assertions (1), (2) and (4) are trivial. We want to prove (3). Note that all fibers of  $\theta$  are irreducible but the compactification  $\overline{\theta}$  may have reducible fiber  $G = \sum G_i$ . Then, by

$$0 > (K_{\overline{X}} + \overline{\Delta}) \cdot G = (K_{\overline{X}} + \overline{\Delta}) \cdot \sum G_i,$$

we obtain  $0 > (K_{\overline{X}} + \overline{\Delta}) \cdot G_i$  for some irreducible component  $G_i$  of the fiber  $G$ . Thus, by Theorem 6.2, we may assume that all fibers of  $\overline{\theta}$  are irreducible. Therefore each fiber  $F$  of  $\overline{\theta}$  is  $(K_{\overline{X}} + \overline{\Delta})$ -negative. It is sufficient to prove that  $F$  generates an extremal ray of  $\overline{NE}(\overline{X})$ . By Theorem 4.4, we have

$$F \equiv D + \sum r_i C_i,$$

where  $D \in \overline{NE}(\overline{X})_{K_{\overline{X}} + \overline{\Delta} \geq 0}$ ,  $r_i \in \mathbb{R}_{\geq 0}$  and each  $C_i$  generates a  $(K_{\overline{X}} + \overline{\Delta})$ -negative extremal ray. Since  $F$  is nef, we have  $F \cdot D = F \cdot C_i = 0$  for all  $i$ . Here recall that all fibers of  $\overline{\theta}$  are irreducible. This means that  $C_i$  is some fiber with the reduced structure. Thus we obtain  $F \equiv qC_i$  for some positive number  $q$  and  $F$  generates an extremal ray.  $\square$

Then, we obtain the minimal model program in full generality.

**Theorem 17.2** (Minimal model Program). *Let  $\pi : X \rightarrow S$  be a projective morphism from a normal surface  $X$  to a variety  $S$ . Let  $\Delta$  be an  $\mathbb{R}$ -divisor on  $X$ . Assume that one of the following conditions holds:*

- (QF)  $X$  is  $\mathbb{Q}$ -factorial and  $0 \leq \Delta \leq 1$ .
- (FP)  $k = \overline{\mathbb{F}}_p$  and  $0 \leq \Delta$ .
- (LC)  $(X, \Delta)$  is a log canonical surface.

*Then, there exists a sequence of proper birational morphisms*

$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\phi_0} (X_1, \Delta_1) \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{s-1}} (X_s, \Delta_s) =: (X^\dagger, \Delta^\dagger)$$

*where  $(\phi_{i-1})_*(\Delta_{i-1}) =: \Delta_i$*

*with the following properties.*

- (1) *Each  $X_i$  is a normal surface, which is projective over  $S$ .*
- (2) *Each  $(X_i, \Delta_i)$  satisfies (QF), (FP) or (LC) according as the above assumption.*
- (3) *For each  $i$ ,  $\text{Ex}(\phi_i) =: C_i$  is a proper irreducible curve such that such that*

$$(K_{X_i} + \Delta_i) \cdot C_i < 0.$$

- (4) Let  $\pi^\dagger : X^\dagger \rightarrow S$  be the  $S$ -scheme structure morphism.  $(X^\dagger, \Delta^\dagger)$  satisfies one of the following conditions.
- (a)  $K_{X^\dagger} + \Delta^\dagger$  is  $\pi^\dagger$ -nef.
  - (b) There is a projective surjective  $S$ -morphism  $\mu : X^\dagger \rightarrow Z$  to a smooth curve  $Z$  such that  $Z$  is projective over  $S$ ,  $\mu_* \mathcal{O}_{X^\dagger} = \mathcal{O}_Z$ ,  $-(K_X^\dagger + \Delta^\dagger)$  is  $\mu$ -ample and  $\rho(X^\dagger/Z) = 1$ .
  - (c)  $X^\dagger$  is a projective surface,  $-(K_{X^\dagger} + \Delta^\dagger)$  is ample and  $\rho(X^\dagger) = 1$ .

In case (a), we say  $(X^\dagger, \Delta^\dagger)$  is a minimal model of  $(X, \Delta)$  over  $S$ .

In case (b) and (c), we say  $(X^\dagger, \Delta^\dagger)$  is a Mori fiber space over  $S$ .

## 18. RELATIVE ABUNDANCE THEOREM

In this section, we consider the relativization of the abundance theorem. To descend the problem from the absolute case to the relative case, let us consider the following lemma.

**Lemma 18.1.** *Let  $\pi : X \rightarrow S$  be a morphism from a projective normal  $\mathbb{Q}$ -factorial surface  $X$  to a projective variety  $S$ . Let  $\Delta$  be an  $\mathbb{R}$ -boundary on  $X$ . If  $K_X + \Delta$  is  $\pi$ -nef, then there exists an ample line bundle  $F$  on  $S$  such that  $\Delta + \pi^*(F) \sim_{\mathbb{R}} \Delta'$  for some  $\mathbb{R}$ -boundary  $\Delta'$  and  $K_X + \Delta'$  is nef.*

*Proof.* Take the Stein factorization of  $\pi$

$$\pi : X \xrightarrow{\theta} T \xrightarrow{\sigma} S.$$

Take an arbitrary ample line bundle  $H$  on  $S$ . Since  $\sigma$  is a finite morphism,  $\sigma^*(H)$  is also ample. We may assume that  $\sigma^*(H)$  is very ample by replacing  $H$  with its multiple. Note that  $\sigma^*(4H)$  is very ample. We want to prove that  $F := 4H$  satisfies the assertion. If  $\dim T = 0$ , then the assertion is obvious. Thus we can consider the following two cases:

(1)  $\dim T = 1$  and (2)  $\dim T = 2$ .

(1) Assume  $\dim T = 1$ . In this case,  $T$  is a smooth projective curve and general fibers of  $\theta$  are integral by Proposition 15.3. Thus, we can take a hyperplane section

$$P_1 + \cdots + P_n = G \in |\sigma^*(4H)|$$

such that  $P_i \neq P_j$  for all  $i \neq j$ ,  $\theta^{-1}(P_i)$  is integral for each  $i$  and  $\theta^{-1}(P_i)$  is not a component of  $\Delta$  for each  $i$ . Therefore, for an  $\mathbb{R}$ -boundary  $\Delta'$  defined by

$$\Delta' := \Delta + \theta^*(G),$$

$K_X + \Delta'$  is nef by Theorem 4.4 and Proposition 4.6.

(2) Assume  $\dim T = 2$ . In this case,  $T$  is a normal projective surface and



$\theta$  is birational. By Bertini's theorem, we can take an irreducible smooth hyperplane section  $G \in |\sigma^*(4H)|$  such that  $\text{Supp} G \cap \theta(\text{Ex}(\theta)) = \emptyset$  and  $G$  is not a component of  $\theta_*(\Delta)$ . Then,  $\Delta' := \Delta + \theta^*(G)$  is an  $\mathbb{R}$ -boundary and  $K_X + \Delta'$  is nef by Theorem 4.4 and Proposition 4.6.  $\square$

We can prove the relative abundance theorem for  $\mathbb{Q}$ -factorial surfaces with  $\mathbb{R}$ -boundary.

**Theorem 18.2.** *Let  $\pi : X \rightarrow S$  be a projective morphism from a normal  $\mathbb{Q}$ -factorial surface  $X$  to a variety  $S$ . Let  $\Delta$  be an  $\mathbb{R}$ -boundary. If  $K_X + \Delta$  is  $\pi$ -nef, then  $K_X + \Delta$  is  $\pi$ -semi-ample.*

*Proof.* We may assume that  $S$  is affine. Moreover, by taking Nagata's compactification we may assume that  $S$  is projective and  $X$  is projective  $\mathbb{Q}$ -factorial. Note that the hypothesis of  $\pi$ -nefness may break up by taking the compactification. But, by running  $(K_X + \Delta)$ -minimal model program over  $S$ , we may assume this hypothesis. Thus we can apply Lemma 18.1.  $F$  and  $\Delta'$  are the same notations as Lemma 18.1. Since  $K_X + \Delta'$  is nef,  $K_X + \Delta'$  is semi-ample by the abundance theorem of the absolute case. By  $K_X + \Delta' \sim_{\mathbb{R}} K_X + \Delta + \pi^*(F)$ ,  $K_X + \Delta$  is  $\pi$ -semi-ample.  $\square$

We obtain the following theorem by applying the same argument.

**Theorem 18.3.** *Let  $\pi : X \rightarrow S$  be a projective morphism from a normal surface  $X$  to a variety  $S$ , which are defined over  $\overline{\mathbb{F}}_p$ . Let  $\Delta$  be an effective  $\mathbb{R}$ -divisor. If  $K_X + \Delta$  is  $\pi$ -nef, then  $K_X + \Delta$  is  $\pi$ -semi-ample.*

*Proof.* We can apply the same proof as Theorem 18.2.  $\square$

The log canonical case immediately follows from the  $\mathbb{Q}$ -factorial case.

**Theorem 18.4.** *Let  $\pi : X \rightarrow S$  be a projective morphism from a log canonical surface  $(X, \Delta)$  to a variety  $S$ . If  $K_X + \Delta$  is  $\pi$ -nef, then  $K_X + \Delta$  is  $\pi$ -semi-ample.*

*Proof.* Take the minimal resolution and apply Theorem 18.2.  $\square$

We summarize the results obtained in this section.

**Corollary 18.5.** *Let  $\pi : X \rightarrow S$  be a projective morphism from a normal surface  $X$  to a variety  $S$ . Let  $\Delta$  be an  $\mathbb{R}$ -divisor on  $X$ . Assume that one of the following conditions holds:*

(QF)  $X$  is  $\mathbb{Q}$ -factorial and  $0 \leq \Delta \leq 1$ .

(FP)  $k = \overline{\mathbb{F}}_p$  and  $0 \leq \Delta$ .

(LC)  $(X, \Delta)$  is a log canonical surface.

*If  $K_X + \Delta$  is  $\pi$ -nef, then  $K_X + \Delta$  is  $\pi$ -semi-ample.*

## Part 6. Appendix

### 19. BASEPOINT FREE THEOREM

In this section, we consider the basepoint free theorem. First, we prove the following non-vanishing theorem.

**Theorem 19.1.** *Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface and let  $\Delta$  be a  $\mathbb{Q}$ -boundary. Let  $D$  be a nef Cartier divisor. Assume that  $D - (K_X + \Delta)$  is nef and big and that  $(D - (K_X + \Delta)) \cdot C > 0$  for every curve  $C \subset \text{Supp } \Delta$ . Then  $\kappa(X, D) \geq 0$ .*

*Proof.* If  $k = \overline{\mathbb{F}}_p$ , then the assertion follows from Theorem 12.6. Thus we may assume  $k \neq \overline{\mathbb{F}}_p$ .

Assume  $\kappa(X, D) = -\infty$  and we derive a contradiction. Let  $f : X' \rightarrow X$  be the minimal resolution,  $K_{X'} + \Delta' = f^*(K_X + \Delta)$  and  $D' = f^*D$ .

**Step 1.** We may assume that  $\kappa(X', K_{X'}) = -\infty$ .

Indeed, we have  $\kappa(X', K_{X'}) \leq \kappa(X', K_{X'} + \Delta') = \kappa(X, K_X + \Delta) = -\infty$ . Note that, if  $\kappa(X, K_X + \Delta) \geq 0$ , then we have  $\kappa(X, D) = \kappa(X, D - (K_X + \Delta) + (K_X + \Delta)) \geq 0$ . This is what we want to show.

**Step 2.** In this step, we show  $h^2(X', D') = 0$ .

By Serre duality, we have

$$h^2(X', D') = h^0(X', K_{X'} - D') \quad \text{and}$$

$$\kappa(X', K_{X'} - D') \leq \kappa(X', K_{X'} + \Delta' - D') = \kappa(X, K_X + \Delta - D) = -\infty,$$

because  $-(K_X + \Delta - D)$  is nef and big.

**Step 3.** In this step, we prove that  $X'$  is an irrational ruled surface.

It is sufficient to prove  $\chi(\mathcal{O}_{X'}) \leq 0$ . Since  $h^0(X', D') = h^2(X', D') = 0$ , by the Riemann–Roch theorem, we obtain

$$-h^1(X', D') = \chi(\mathcal{O}_{X'}) + \frac{1}{2}D' \cdot (D' - K_{X'}).$$

Since

$$D' \cdot (D' - K_{X'}) = D \cdot (D - K_X) \quad \text{and}$$

$$\kappa(X, D - K_X) \geq \kappa(X, D - (K_X + \Delta)) = 2,$$

we have  $D' \cdot (D' - K_{X'}) \geq 0$  by the nefness of  $D$ . Therefore we get  $0 \geq -h^1(X', D') \geq \chi(\mathcal{O}_{X'})$ .

Let  $\pi : X' \rightarrow Z$  be its ruling. By Theorem 6.1,  $\pi$  factors through  $X$ .

**Step 4.** We reduce the proof to the case where there is no curve  $C$  in  $X$  such that  $D \cdot C = 0$  and  $C^2 < 0$ .

Let  $C$  be such a curve. We have  $(K_X + C) \cdot C < 0$  by the assumption. (Indeed, if  $C \subset \text{Supp } \Delta$ , then

$$-(K_X + C) \cdot C \geq -(K_X + \Delta) \cdot C = (D - (K_X + \Delta)) \cdot C > 0.$$

If  $C \not\subset \text{Supp } \Delta$ , then

$$-(K_X + C) \cdot C > -(K_X + \Delta) \cdot C = (D - (K_X + \Delta)) \cdot C \geq 0.)$$

This shows that  $C = \mathbb{P}^1$  and  $C$  is contractable. Moreover this induces a contraction map  $g : X \rightarrow Y$  to a  $\mathbb{Q}$ -factorial surface  $Y$  and the irrationality of  $X$  shows that  $\pi$  factors through  $Y$ . Let  $g_*D = D_Y$  and  $g_*(\Delta) = \Delta_Y$ . Then, we have  $K_X + \Delta = g^*(K_Y + \Delta_Y) + aC$  for some non-negative rational number  $a$ . Therefore, it is easy to see that  $Y$  has all the assumptions of  $X$ .

**Step 5.** We reduce the proof to the case where  $K_X + \Delta$  is not nef. In particular, there is at least one  $(K_X + \Delta)$ -negative extremal ray.

If  $K_X + \Delta$  is nef, then  $D = D - (K_X + \Delta) + (K_X + \Delta)$  is nef and big, and this is what we want to show. Thus, we may assume that  $K_X + \Delta$  is not nef.

**Step 6.** We reduce the proof to the case where  $D \equiv 0$ .

The nefness of  $D$  and  $\kappa(X, D) = -\infty$  show  $D^2 = 0$ . Since  $D$  and  $D - (K_X + \Delta)$  is nef, we have  $(D - (K_X + \Delta)) \cdot D = -(K_X + \Delta) \cdot D \geq 0$ . We consider the two cases:  $-(K_X + \Delta) \cdot D = 0$  or  $-(K_X + \Delta) \cdot D > 0$ . If  $-(K_X + \Delta) \cdot D = 0$ , then we obtain  $D \equiv 0$  by the bigness of  $D - (K_X + \Delta)$ . This is what we want to show. If  $-(K_X + \Delta) \cdot D > 0$ , then we have  $K_X \cdot D < 0$ . Two conditions  $K_X \cdot D < 0$  and  $D^2 = 0$  mean  $\kappa(X, D) = 1$  by resolution and the Riemann–Roch theorem. This case is excluded.

**Step 7.** By Step 4 and Step 6, there exists no curve  $C$  with  $C^2 < 0$ . By Step 5 and the classification of extremal rays, we have  $\rho(X) \leq 2$ . Since there is a surjection  $X \rightarrow Z$  to a curve  $Z$ , we have  $\rho(X) \neq 1$ . Thus, we obtain  $\rho(X) = 2$ . Here,  $-(K_X + \Delta)$  is ample because  $-(K_X + \Delta)$  is nef and big and Step 4. Moreover, by Step 4, there are two extremal rays inducing the structure of the Mori fiber space to a curve. By Proposition 4.6, every extremal ray is generated by a rational curve. This contradicts the irrationality of  $Z$ .

This completes the proof.  $\square$

Using the non-vanishing theorem, we obtain the following basepoint free theorem.

**Theorem 19.2.** *Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface and let  $\Delta$  be a  $\mathbb{Q}$ -boundary. Let  $D$  be a nef Cartier divisor. Assume that*

$D - (K_X + \Delta)$  is nef and big and that  $(D - (K_X + \Delta)) \cdot C > 0$  for every curve  $C \subset \text{Supp } \Delta$ . Then  $D$  is semi-ample.

*Proof.* By Theorem 19.1, we may assume  $\kappa(X, D) \geq 0$ . But by Proposition 6.4, we may assume  $\kappa(X, D) = 0$  or 2. By the same argument as Step 4 in the proof of Theorem 19.1, we may assume that there is no curve  $C$  in  $X$  with  $D \cdot C = 0$  and  $C^2 < 0$ . Thus, if  $\kappa(X, D) = 2$ , then  $D$  is ample. This is what we want to show. Hence the remaining case is  $\kappa(X, D) = 0$ . We have linear equivalence to effective divisor  $nD \sim \sum d_i D_i$ . Assume  $\sum d_i D_i \neq 0$  and let us get a contradiction. Since  $D^2 = 0$  and the nefness of  $D$ , we have  $D \cdot D_i = 0$  for all  $i$ . Moreover we get  $D_i^2 \geq 0$  by the above reduction. Then, we obtain  $D_i^2 = D_i \cdot D = 0$ . Since  $D - (K_X + \Delta)$  is nef and big, we have

$$(D - (K_X + \Delta)) \cdot D_i = -(K_X + \Delta) \cdot D_i > 0.$$

This means  $K_X \cdot D_i < 0$ .  $D_i^2 = 0$  and  $K_X \cdot D_i < 0$  show  $\kappa(X, D_i) = 1$  by taking a resolution and applying the Riemann–Roch theorem. This contradicts  $\kappa(X, D) = 0$ .  $\square$

The following example teaches us that the basepoint free theorem does not hold only under the boundary condition.

**Example 19.3.** *If  $k \neq \overline{\mathbb{F}}_p$ , then there exist a smooth projective surface  $X$  over  $k$ , an elliptic curve  $C$  in  $X$  and a divisor  $D$  such that  $K_X + C = 0$  and the divisor  $D = D - (K_X + C)$  is nef and big but not semi-ample.*

*Construction.* Let  $X_0 := \mathbb{P}^2$  and let  $C_0$  be an elliptic curve in  $X_0$ . Let  $P_1, \dots, P_{10}$  be ten points which are linearly independent. Blowup these 10 points. We obtain the surface  $X$  and let  $C$  be the proper transform of  $C_0$ . Then  $K_X + C = 0$  and  $C$  is not contractable by Answer 10.2 and its construction. On the other hand, take an ample divisor  $H$  and let  $D$  be the divisor  $D := H + qC$  with  $(H + qC) \cdot C = 0$ . It is easy to check that  $D$  is nef and big. Because  $C$  is not contractable,  $D$  is not semi-ample.  $\square$

We can also prove a basepoint free theorem under the following assumption.

**Theorem 19.4.** *Let  $X$  be a projective normal  $\mathbb{Q}$ -factorial surface and let  $\Delta$  be a  $\mathbb{Q}$ -boundary. Let  $D$  be a nef Cartier divisor. Assume that  $D - (K_X + \Delta)$  is semi-ample. Then  $D$  is semi-ample.*

*Proof.* Set  $\kappa := \kappa(X, D - (K_X + \Delta))$ . There are three cases: (0)  $\kappa = 0$ , (1)  $\kappa = 1$  and (2)  $\kappa = 2$ .

(0) Assume  $\kappa = 0$ . By the semi-ampleness, we obtain  $D - (K_X + \Delta) \sim_{\mathbb{Q}} 0$ . Thus we can apply the abundance theorem to  $D$ . Then we obtain

the desired result.

(1) Assume  $\kappa = 1$ . By the semi-ampleness, the complete linear system  $|n(D - (K_X + \Delta))|$  induces a morphism  $\sigma : X \rightarrow B$  to a smooth projective curve. By Proposition 15.3, we can find a boundary

$$\Delta' \sim_{\mathbb{Q}} D - (K_X + \Delta)$$

such that  $\Delta + \Delta'$  is a  $\mathbb{Q}$ -boundary. Thus we can apply the abundance theorem to  $K_X + \Delta + \Delta'$ .

(2) Assume  $\kappa = 2$ . The complete linear system  $|n(D - (K_X + \Delta))|$  induces a birational morphism  $f : X \rightarrow Y$  to a normal projective surface. Since  $n(D - (K_X + \Delta)) = f^*(H_Y)$  where  $H_Y$  is a very ample line bundle on  $Y$ . By Bertini's theorem, we can find a member  $G \in |H_Y|$  such that

$$\Delta + \frac{1}{n}f^*(G)$$

is a boundary. Thus we can apply the abundance theorem.  $\square$

## 20. RATIONAL SINGULARITIES

In this section, we consider the relation between the minimal model program and the rational singularities.

**Definition 20.1.** Let  $X$  be a normal surface and let  $f : Y \rightarrow X$  be a resolution of singularities. We say  $X$  has at worst rational singularities if  $R^1f_*\mathcal{O}_Y = 0$ . This property is independent of the choice of resolutions of singularities.

If  $X$  is a normal surface whose singularities are at worst rational, then  $X$  is  $\mathbb{Q}$ -factorial by [Lipman, Proposition 17.1]. Let us give an alternative proof of this result.

**Proposition 20.2.** *Let  $X$  be a normal surface. If  $X$  has at worst rational singularities, then  $X$  is  $\mathbb{Q}$ -factorial.*

*Proof.* Note that, if  $g : Z \rightarrow X$  is a proper birational morphism and  $E$  is a  $g$ -exceptional curve, then  $E \simeq \mathbb{P}^1$ .

We may assume  $X$  is affine. Thus, we may assume  $X$  is projective. Let  $f : Y \rightarrow X$  be the minimal resolution. Let  $E$  be an  $f$ -exceptional curve. By Proposition 6.5, we can contract  $E$  and we obtain

$$f : Y \rightarrow Y' \xrightarrow{f'} X.$$

By Proposition 6.6 and Proposition 6.7,  $X'$  is  $\mathbb{Q}$ -factorial. Assume  $f'$  is not an isomorphism. Then, we can take an  $f'$ -exceptional curve  $E'$ . By the same argument, we can contract  $E'$  to a  $\mathbb{Q}$ -factorial surface. Repeat the same procedure. Then, we see  $X$  is  $\mathbb{Q}$ -factorial.  $\square$

The Kodaira vanishing theorem do not hold in positive characteristic. But we obtain the following relative vanishing theorem.

**Theorem 20.3.** *Let  $f : X \rightarrow Y$  be a proper birational morphism from a smooth surface  $X$  to a normal surface  $Y$ . Let  $L$  be a line bundle on  $X$  such that*

$$L \equiv_f K_X + E + N$$

*where  $E$  is an effective  $f$ -exceptional  $\mathbb{R}$ -boundary and  $N$  is an  $f$ -nef  $\mathbb{R}$ -divisor. If  $E_i \cdot N > 0$  for every curve  $E_i$  with  $E_i \subset \lfloor E \rfloor$ , then  $R^1 f_*(L) = 0$ .*

*Proof.* See [Kollár-Kovács, Section 2.2].  $\square$

In this paper, we often use the contraction of  $\mathbb{P}^1$ . For example, the minimal model program of Theorem 6.8 is the composition of the contractions of  $C \simeq \mathbb{P}^1$  with  $(K_X + C) \cdot C < 0$ . The following theorem shows that  $R^1$  of such contractions vanish.

**Theorem 20.4.** *Let  $g : Y \rightarrow Z$  be a proper birational morphism between normal surfaces such that  $C := \text{Ex}(g)$  is an irreducible curve. If  $(K_Y + C) \cdot C < 0$ , then  $R^1 g_*(\mathcal{O}_Y) = 0$ .*

*Proof.* Let  $f : X \rightarrow Y$  be the minimal resolution of  $Y$  and let  $C_X$  be the proper transform of  $C$ . Set  $K_X + C_X + \Delta_X = f^*(K_Y + C)$ . Then we have

$$-\lfloor \Delta_X \rfloor = K_X + (\{\Delta_X\} + C_X) - f^*(K_Y + C).$$

We apply Theorem 20.3 and we obtain

$$R^1(g \circ f)_*(-\lfloor \Delta_X \rfloor) = 0$$

by  $-f^*(K_Y + C) \cdot C_X > 0$ .

If  $\lfloor \Delta_X \rfloor = 0$ , then we obtain

$$R^1 g_*(\mathcal{O}_Y) = R^1 g_*(f_* \mathcal{O}_X) \subset R^1(g \circ f)_*(\mathcal{O}_X) = 0$$

by the Grothendieck–Leray spectral sequence. Thus we may assume  $\lfloor \Delta_X \rfloor \neq 0$ . Since

$$0 \rightarrow \mathcal{O}_X(-\lfloor \Delta_X \rfloor) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\lfloor \Delta_X \rfloor} \rightarrow 0,$$

we obtain

$$\begin{aligned} 0 \rightarrow f_* \mathcal{O}_X(-\lfloor \Delta_X \rfloor) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{C} \rightarrow 0 \\ \mathcal{C} \subset f_* \mathcal{O}_{\lfloor \Delta_X \rfloor} \end{aligned}$$

where  $\mathcal{C}$  is the cokernel of  $f_* \mathcal{O}_X(-\lfloor \Delta_X \rfloor) \rightarrow \mathcal{O}_Y$ . Since  $f_* \mathcal{O}_{\lfloor \Delta_X \rfloor}$  is a skyscraper sheaf, so is  $\mathcal{C}$ . Thus we obtain

$$R^1 g_*(f_* \mathcal{O}_X(-\lfloor \Delta_X \rfloor)) \rightarrow R^1 g_*(\mathcal{O}_Y) \rightarrow R^1 g_*(\mathcal{C}) = 0.$$

By the Grothendieck–Leray spectral sequence, we obtain

$$R^1 g_*(f_* \mathcal{O}_X(-\lfloor \Delta_X \rfloor)) \subset R^1(g \circ f)_* \mathcal{O}_X(-\lfloor \Delta_X \rfloor) = 0.$$

Therefore, we have  $R^1 g_*(\mathcal{O}_Y) = 0$ .  $\square$

As corollaries, we obtain the results on minimal models and canonical models for surfaces with rational singularities.

**Corollary 20.5.** *Let  $\pi : X \rightarrow S$  be a projective morphism from a normal surface  $X$  to a variety  $S$ . Let  $\Delta$  be an  $\mathbb{R}$ -boundary. Assume  $X$  has at worst rational singularities. Then, the following assertions holds.*

- (1)  *$X$  is  $\mathbb{Q}$ -factorial. In particular, by Theorem 17.2, we can run a  $(K_X + \Delta)$ -minimal model program*

$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\phi_0} (X_1, \Delta_1) \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{s-1}} (X_s, \Delta_s)$$

*where  $(\phi_{i-1})_*(\Delta_{i-1}) =: \Delta_i$ .*

- (2) *Each  $X_i$  has at worst rational singularities.*

*Proof.* (1) follows from Proposition 20.2. Each extremal contraction in a minimal model program of  $(X, \Delta)$  satisfies the condition of Theorem 20.4. This implies (2).  $\square$

**Corollary 20.6.** *Let  $\pi : X \rightarrow S$  be a projective morphism from a normal surface  $X$  to a variety  $S$ . Let  $\Delta$  be an  $\mathbb{R}$ -divisor such that  $0 \leq \Delta < 1$ . If  $X$  has at worst rational singularities and  $K_X + \Delta$  is  $\pi$ -big, then the canonical model of  $(X, \Delta)$  over  $S$  has at worst rational singularities.*

*Proof.* By Corollary 20.5 we may assume that  $K_X + \Delta$  is  $\pi$ -nef and  $\pi$ -big. If  $(K_X + \Delta) \cdot C = 0$  for some curve  $C$  such that  $\pi(C)$  is one point, then  $(K_X + C) \cdot C < 0$  because  $0 \leq \Delta < 1$ . Therefore we can contract this curve  $C$  and  $C$  satisfies the condition of Theorem 20.4. Repeat this procedure and we obtain the required assertion.  $\square$

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